
Exercise 1: Unequal time commutator

Consider a real quantized scalar field $\hat{\phi}(x)$ with conjugate field $\hat{\pi}(y)$, where $x^0 \neq y^0$. Show that the unequal time commutator is given as

$$[\hat{\phi}(x), \hat{\pi}(y)] = \frac{\mathrm{i}}{2} \int \frac{d^3p}{(2\pi)^3} \left(e^{\mathrm{i}p\cdot(x-y)} + e^{-\mathrm{i}p\cdot(x-y)} \right), \tag{1}$$

by using the Fourier transform of $\hat{\phi}$ and $\hat{\pi}$, familiar from the lecture.
Show that this reduces to the equal time commutator, when choosing $x^0 = y^0 = t$.

Exercise 2: Occupation number

Prove the following relation of the number operator $\hat{N}$

$$\hat{N}\,|\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, ..., \mathbf{k}^{(n)}\rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})\,|\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, ..., \mathbf{k}^{(n)}\rangle = n\,|\mathbf{k}^{(1)}, \mathbf{k}^{(2)}, ..., \mathbf{k}^{(n)}\rangle\,. \tag{2}$$

Hint: Use induction.

Exercise 3: The four momentum operator of electrodynamics

In the lecture you have discussed the quantized four potential of electrodynamics

$$\hat{A}^\mu(x) = \int \sum_{\lambda=1}^{2} \epsilon^\mu_\lambda(\mathbf{p}) \left(\hat{a}_\lambda(\mathbf{p}) e^{-ipx} + \hat{a}^\dagger_\lambda(\mathbf{p}) e^{ipx} \right) \frac{d^3p}{(2\pi)^3 2E(\mathbf{p})}. \tag{3}$$

With the polarization vectors $\epsilon^\mu_\lambda(\mathbf{p})$ and dispersion relation $\omega = E(\mathbf{p}) = |\mathbf{p}|$. Furthermore let us consider radiation gauge

$$\hat{A}^0 = 0, \qquad \nabla \hat{\mathbf{A}} = 0, \tag{4}$$

making it possible to choose two real linear independent polarization vectors that are normalizable

$$\epsilon^\mu_\lambda \epsilon_{\mu,\lambda'} = \epsilon^i_\lambda \epsilon_{i,\lambda'} = -\delta_{\lambda\lambda'}. \tag{5}$$

i) Calculate the normal ordered zero component of the four momentum operator

$$\hat{P}^0 = \int \hat{\mathcal{H}} d^3x = \frac{1}{2} \int (|\hat{\mathbf{E}}|^2 + |\hat{\mathbf{B}}|^2) d^3x \tag{6}$$

Hint: Use $|\hat{\mathbf{E}}|^2 = \hat{F}^{i0}\hat{F}_{0i}$ *and* $|\hat{\mathbf{B}}|^2 = \frac{1}{2}\hat{F}^{ij}\hat{F}_{ij}$.

ii) An analogous calculation can be done for the spatial components of the four momentum operator, leading to

$$\hat{P}^i = \int \left(\hat{\mathbf{E}} \times \hat{\mathbf{B}}\right)^i d^3x = \frac{1}{2}\int p^i \sum_{\lambda=1}^{2} \hat{a}^\dagger_\lambda(\mathbf{p})\hat{a}_\lambda(\mathbf{p})\frac{d^3p}{(2\pi)^3 2E(\mathbf{p})}, \tag{7}$$

making it possible to give a normal ordered expression of the complete four momentum operator

$$\hat{P}^\mu = \int \frac{p^\mu}{E(\mathbf{p})} \sum_{\lambda=1}^{2} \hat{a}^\dagger_\lambda(\mathbf{p})\hat{a}_\lambda(\mathbf{p})\frac{d^3p}{2(2\pi)^3}. \tag{8}$$

Prove that the first excited Fock state is an eigenstate of the four momentum operator

$$\hat{P}^\mu \left|a_\lambda(\mathbf{p})\right\rangle = \hat{P}^\mu \hat{a}^\dagger_\lambda(\mathbf{p}) \left|0\right\rangle = p^\mu \left|a_\lambda(\mathbf{p})\right\rangle. \tag{9}$$