
Exercise 1: Complex scalar field

Consider the action of a complex scalar field

$$S[\phi, \phi^*, \partial^\mu \phi, \partial^\mu \phi^*] = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi). \tag{1}$$

i) Treating ϕ and ϕ^* as dynamical variables, find their conjugate momenta π and π^* and show that the Hamiltonian has the following form

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi). \tag{2}$$

ii) Show hat the Lagrange density is invariant with respect to the following (global) transformation

$$\phi(x) \to \phi'(x) = e^{-\mathrm{i}\alpha} \phi(x), \qquad \phi^*(x) \to \phi^{*\prime} = e^{\mathrm{i}\alpha} \phi^*(x), \tag{3}$$

with α being a constant. Calculate the associated Noether current

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\delta \phi^*}{\delta \alpha} \tag{4}$$

and show that it is conserved

$$\partial_\mu j^\mu = 0. \tag{5}$$

iii) The energy momentum tensor of a theory with $n \in \mathbb{N}$ scalar fields ϕ_n, described by a Lagrange density $\mathcal{L} = \mathcal{L}(\phi_n, \partial_\mu \phi_n)$ is given as

$$\Theta^{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial^\nu \phi_n - g^{\mu\nu} \mathcal{L}. \tag{6}$$

Treating ϕ and ϕ^* as dynamical variables calculate the energy momentum tensor for the theory of a complex scalar field and show that it is symmetric $\Theta^{\mu\nu} = \Theta^{\nu\mu}$.

Exercise 2: The energy momentum tensor in Electrodynamics

Consider the Lagrange density of classical electrodynamics without external currents or charges

$$\mathcal{L}(A^\nu, \partial^\mu A^\nu) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{7}$$

with the electromagnetic field-strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{8}$$

i) Calculate the associated energy-momentum tensor

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}. \tag{9}$$

ii) Show that it is neither symmetric $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$, nor gauge invariant under a gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu f(x). \tag{10}$$

iii) Our aim is now to construct the symmetric and gauge-invariant energy momentum tensor of electrodynamics. For this purpose we add the following quantity

$$\tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \partial_\beta K^{\beta\mu\nu}, \tag{11}$$

with $K^{\beta\mu\nu}$ being antisymmetric in β and μ. To prove that the additional term does not change energy momentum conservation, show that the energy momentum tensor is conserved

$$\partial_\mu \tilde{\Theta}^{\mu\nu} = 0. \tag{12}$$

Finally show that the choice

$$K^{\beta\mu\nu} = F^{\mu\beta} A^\nu \tag{13}$$

leads to a symmetric and gauge-invariant energy momentum tensor $\tilde{\Theta}^{\mu\nu}$.

iv) Show that the zero component of $\tilde{\Theta}^{00}$ reproduces indeed the energy density familiar from electrodynamics

$$\tilde{\Theta}^{00} = \epsilon = \frac{1}{2} \left(|\mathbf{E}|^2 + |\mathbf{B}|^2 \right), \tag{14}$$

and the $\tilde{\Theta}^{0i}$ components give the Poynting vector

$$\tilde{\Theta}^{0i} = \mathbf{S}^i = (\mathbf{E} \times \mathbf{B})^i. \tag{15}$$