
Exercise 1: Left- and right-handed projection operators

We define the following projection operators, projecting on the left- and right-handed components of the Dirac spinors

$$P_L = \frac{\mathbb{1} - \gamma^5}{2} \to \psi_L := P_L\psi, \qquad P_R = \frac{\mathbb{1} + \gamma^5}{2} \to \psi_R := P_R\psi. \tag{1}$$

i) Show the following properties of projection operators

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0. \tag{2}$$

ii) The helicity is defined as the projection of the spin $\mathbf{S}$ of a particle on its momentum $\mathbf{p}$

$$h(\mathbf{p}) := \frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|} = \frac{1}{2|\mathbf{p}|}\begin{pmatrix} \sigma\cdot\mathbf{p} & 0 \\ 0 & \sigma\cdot\mathbf{p} \end{pmatrix}. \tag{3}$$

Show that in case of the massless Dirac equation, the left- and right-handed spinors defined by $P_L u_s(\mathbf{p})$ and $P_L v_s(\mathbf{p})$ are eigenstates of the helicity operator and determine the eigenvalues. Does this also hold for $m \neq 0$?

Exercise 2: Covariant Electrodynamics

The Lagrange density of Electrodynamics is given as

$$\mathscr{L}(A^\mu(x), \partial^\mu A^\nu(x), t) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu(x) \tag{4}$$

with $x = x^\mu = (t, \mathbf{x})$, $A^\mu(x) = (\phi(x), \mathbf{A}(x))$, where $\phi(x)$ is the scalar- and $\mathbf{A}(x)$ the vector-potential. The current is given as $j^\mu = (\rho(x), \mathbf{j}(x))$, where $\rho(x)$ is the distribution of electric charge and $\mathbf{j}(x)$ the electric current. The electromagnetic tensor is defined as

$$F^{\mu\nu} := \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \tag{5}$$

Use the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathscr{L}}{\partial A_\nu} = 0 \tag{6}$$

to derive the covariant inhomogenous Maxwell equations.

Exercise 3: Functional derivative

Consider the following 1-dimensional action

$$S[x] = \int_{t_i}^{t_f} L(x(t'), \dot{x}(t'))dt', \tag{7}$$

with Lagrange function

$$L(x(t), \dot{x}(t)) = \frac{1}{2}m(\dot{x}(t))^2 - V(x(t)). \tag{8}$$

Calculate the functional derivative $\frac{\delta S[x]}{\delta x(t)}$ for $t_i \leq t \leq t_f$.