
Exercise 1: QCD running coupling

In the lecture you discussed that $\alpha_s = \frac{g^2}{4\pi}$, with g the strong coupling of QCD, is a function of energy. This running coupling is given by the following differential equation,

$$Q^2 \frac{d\alpha_s}{dQ^2} = -\beta_0 \alpha_s^2(Q^2) + \mathcal{O}(\alpha_s^3), \tag{1}$$

where Q^2 is the energy scale of the system and β_0 is given as

$$\beta_0 = \frac{11N_c - 2N_f}{12\pi}, \tag{2}$$

with N_c colors and N_f flavors.

i) Solve the differential equation up to first order, ignoring all terms $\mathcal{O}(\alpha_s^3)$ and higher.

ii) The value for α_s at the energy scale of the mass of the Z-boson, $Q = M_Z = 91.1$ GeV is

$$\alpha_s(Q^2 = M_Z^2) = 0.12. \tag{3}$$

Use $Q = 10$ GeV with $N_f = 5$ flavors and calculate the value of α_s.

Exercise 2: Fermion mass generation

Consider the following theory for real scalar fields and fermions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \bar{\psi}\mathrm{i}\gamma^\mu\partial_\mu\psi - g\phi\bar{\psi}\psi, \tag{4}$$

with the following potential

$$V(\phi) = \frac{\lambda}{4!}\left(\phi^2 - v^2\right)^2. \tag{5}$$

i) Sketch the potential $V(\phi)$ of the scalar field. What is the vacuum expectation value $\langle\phi\rangle = \langle 0|\,\phi\,|0\rangle$?

ii) Reparametrise the scalar field by its fluctuations around the expectation value $\phi = \langle\phi\rangle + \chi$ and express $\mathcal{L}$ in terms of the field variables ψ and χ.

iii) Identify the particle masses m_ψ and m_ξ and the basic vertices for all interaction terms in this theory, without specifying the Feynman rules. Is there a Goldstone boson? Discuss without calculations what changes in case of a complex scalar field.

******************************* **Bonus** *********************************

Exercise 3: Electroweak theory

Consider the following terms of the electroweak Lagrangian for the first generation of leptons

$$\mathcal{L}_f = \bar{\ell}_L \mathrm{i}\gamma^\mu \tilde{D}_\mu \ell_L + \bar{e}_R \mathrm{i}\gamma^\mu D_\mu e_R, \tag{6}$$

with $\ell_L = (\nu_L, e_L)$ the left-handed $SU(2)$ isospin doublet and e_R the right-handed singlet. The covariant derivatives are given as

$$\left(\tilde{D}_\mu\right)_{bc} = \partial_\mu \delta_{bc} + \frac{\mathrm{i}g'}{2} B_\mu \delta_{bc} - \mathrm{i}g T^a_{bc} W^a_\mu, \tag{7}$$

$$D_\mu = \partial_\mu + \mathrm{i}g' B_\mu, \tag{8}$$

where T^a are the generators of $SU(2)$.

i) Insert the covariant derivatives and simplify as much as possible to identify the kinetic and interaction terms.

ii) Identify the currents

$$j^\mu_{\nu_L} = \bar{\nu}_L \gamma^\mu \nu_L, \quad j^\mu_{e_L} = \bar{e}_L \gamma^\mu e_L, \quad j^\mu_{e_R} = \bar{e}_R \gamma^\mu e_R, \tag{9}$$

to rewrite the interaction terms and replace the coupling g' using the Weinberg mixing angle

$$\tan\theta_W = \frac{g'}{g}. \tag{10}$$

iii) The gauge fields B_μ and W_μ are connected to the physical gauge bosons $\{W^\pm, Z, A\}$. Use the following redefinitions and rotations to rewrite the Lagrangian in terms of physical fields

$$W^+_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu - \mathrm{i}W^2_\mu\right), \tag{11}$$

$$W^-_\mu = \frac{1}{\sqrt{2}}\left(W^1_\mu + \mathrm{i}W^2_\mu\right), \tag{12}$$

$$B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu, \tag{13}$$

$$W^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu. \tag{14}$$

Identify the electrical charge e as a function of θ_W and g.