
Exercise 1: Quark-gluon scattering

Consider the theory of massless QCD with one quark flavour, denoted q

$$\mathcal{L} = \bar{q}\left(\mathrm{i}\gamma^\mu D_\mu\right) q - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}. \tag{1}$$

Here $D_\mu = \partial_\mu - \mathrm{i}gT^a A^a_\mu$ is the covariant derivative with $SU(3)$ gauge fields A^a_μ describing gluons, and the generators T^a of $SU(3)$.
Let us have a look at quark gluon scattering

$$q(p, r) + g(k) \rightarrow q(p', s) + g(k'), \tag{2}$$

with incoming momenta p, k and outgoing momenta p', k' and spins r, s.

i) Identify all contributing Feynman diagrams up to order $\mathcal{O}(g^2)$. (Self interaction!)

ii) The relevant QCD Feynman rules in this case are

- A quark-gluon vertex has a contribution of

$$\sim \mathrm{i}g\gamma_\mu T^a_{ij}. \tag{3}$$

- A gluon three-vertex, has a contribution of

$$\sim g f_{abc}\left(g_{\mu\nu}(p_1 - p_2)_\rho + g_{\nu\rho}(p_2 - p_3)_\mu + g_{\rho\mu}(p_3 - p_1)_\nu\right), \tag{4}$$

  where by convention all momenta p_i of the gluons are denoted as incoming.

- An internal quark line gets a factor of

$$\sim \mathrm{i}\frac{\not{p}\delta_{ij}}{p^2 + \mathrm{i}\epsilon}, \tag{5}$$

  with δ_{ij} being a Kronecker delta for color indices $i, j \in \{1, ..., N\}$ of the fundamental representation.

- An internal gluon line (in Feynman gauge $\xi = 1$) gets a factor of

$$\sim -\mathrm{i}\frac{g_{\mu\nu}\delta_{ab}}{p^2 + \mathrm{i}\epsilon}, \tag{6}$$

  with δ_{ab} being a Kronecker delta for color indices $a, b \in \{1, ..., N^2 - 1\}$ of the adjoint representation (gluons).

- An incoming/outgoing quark gets a factor of $u_{i,r}(p)/\bar{u}_{i,r}(p)$, an incoming/outgoing gluon a factor of $\epsilon^{\mu a}_\lambda(k)/\epsilon^{\star\nu a}_\lambda$.

Give the matrix element of each Feynman diagram from i) using the given Feynman rules.
Hint: Use Mandelstam variables.

Exercise 2: Vektor and axial flavour symmetry of QCD

Consider the quark sector of the QCD Lagrangian with N_f flavours $\psi_f = (\psi_1, \psi_2, ..., \psi_{N_f})$

$$\mathcal{L} = \sum_{f=1}^{N_f} \bar{\psi}_f (\mathrm{i}\gamma^\mu D_\mu - m_f)\psi_f.$$

i) Show that in case of degenerate quark masses, $m_f = m$ with $f \in \{1, ..., N_f\}$, the theory is invariant under the following global flavour transformation,

$$\psi \to \psi' = U\psi = e^{-\mathrm{i}\theta^a T^a}\psi,$$

with $U \in SU(N_f)$, constant parameters θ^a and the generators T^a of the $SU(N_f)$ group.

ii) Consider the following axial flavour transformation of Dirac spinors

$$\psi \to \psi' = U\psi = e^{-\mathrm{i}\omega^a T^a \gamma_5}\psi.$$

Determine first the corresponding transformation of $\bar{\psi}$, and then show that the Lagrangian is invariant only if $m = 0$

Exercise 3: Quark-antiquark scattering

Quarks carry electric charge as well as color charge and hence interact both via the electromagnetic as well as the strong interaction.

i) Draw all tree level Feynman diagrams for quark-antiquark scattering of an up and an antidown quark

$$u + \bar{d} \to u + \bar{d} \tag{7}$$

both for QED and QCD.

ii) Write down the corresponding scattering amplitudes $\mathrm{i}\mathcal{M}_{QED}$ and $\mathrm{i}\mathcal{M}_{QCD}$. Use the QCD Feynman rules in momentum space familiar from exercise 1 for the latter one.

iii) The spin sums of QCD in the case of massless quarks take the following form

$$\sum_s u_s^i(\mathbf{p})\bar{u}_s^j(\mathbf{p}) = \not{p}\delta^{ij}, \quad \sum_s v_s^i(\mathbf{p})\bar{v}_s^j(\mathbf{p}) = \not{p}\delta^{ij} \tag{8}$$

with $i, j \in \{1, ...3\}$ the group indices of the fundamental representation of the gauge group $SU(3)$. Use these sums to average over initial spins and colors and sum over final spins and colors to relate the scattering amplitude of QCD to the scattering amplitude of QED

$$|\mathcal{M}_{QCD}|^2 = R|\mathcal{M}_{QED}|^2. \tag{9}$$

Determine the factor R as a function of strong coupling g_s and the QED quark charges q_u and q_d.

Hint: You do not have to carry out any traces of gamma matrices after performing the spin sums. It is sufficient to look at the Kronecker deltas and group generators of the $SU(3)$ gauge group to evaluate R.