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**Exercise 1: Adjoint Representation of $SU(N)$**

Consider the Lie group $SU(N)$. The fundamental representation of the group is given by the $N \times N$ matrices $U \in SU(N)$

$$U = e^{-\mathrm{i}\theta^a T^a}, \qquad a = 1, ..., N^2 - 1, \tag{1}$$

with generators $T^a$ of the Lie group and antisymmetric structure constants $f^{abc}$, that satisfy the Lie algebra

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c, \tag{2}$$

and are normalized as

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}. \tag{3}$$

A field $\phi^a$ is said to transform with the adjoint representation of the group $SU(N)$ if the associated matrix valued field, defined by $\phi := \phi^a T^a$ transforms as

$$\phi' = U \phi U^\dagger. \tag{4}$$

i) Show that the transformation of the components $\phi^a$, and therefore the adjoint representation $D^{ab}(U)$ is given as

$$\phi^a \to \phi^{a\prime} = D^{ab}(U)\phi^b = 2\mathrm{tr}(U^\dagger T^a U T^b)\phi^b. \tag{5}$$

ii) The covariant derivative acting on the matrix valued field $\phi = \phi^a T^a$ is defined as

$$D_\mu \phi(x) = \partial_\mu \phi(x) - \mathrm{i}g[A_\mu, \phi(x)]. \tag{6}$$

Show that this indeed transforms as the adjoint representation

$$(D_\mu \phi)'(x) = U(x)(D_\mu \phi(x))U^\dagger(x), \tag{7}$$

with $U(x) \in SU(N)$ and that

$$(D_\mu \phi)^a = \partial_\mu \phi^a - g f^{abc} A^c_\mu \phi^b. \tag{8}$$

iii) Prove that for two matrices $U, U' \in SU(N)$ in fundamental representation, one has

$$D^{ab}(U) D^{bc}(U') = D^{ac}(UU'), \tag{9}$$

by making use of the following relation

$$T^a_{li} T^a_{no} = \frac{1}{2}\left(\delta_{lo}\delta_{in} - \frac{1}{N}\delta_{li}\delta_{no}\right). \tag{10}$$

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**Exercise 2: Yang-Mills theory**

In the lecture you derived the covariant derivative of $SU(N)$ gauge theory

$$D_\mu = \partial_\mu - \mathrm{i} g A_\mu^a T^a, \tag{11}$$

with $SU(N)$ gauge field $A_\mu^a$ and the generators $T^a$ of the $SU(N)$ group, normalized to $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

i) Use the covariant derivative to calculate the field strength tensor $F_{\mu\nu}^a$ of $SU(N)$ gauge theory via

$$F_{\mu\nu} = \frac{\mathrm{i}}{g}[D_\mu, D_\nu]. \tag{12}$$

ii) The Lagrangian of Yang-Mills theory is constructed from the field strength tensor $F^{\mu\nu}$ in the following, gauge invariant way

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right). \tag{13}$$

Use your result from i) to express the Lagrangian in terms of $A^{\mu a}$ and identify the self interaction terms.

**Exercise 3: Isospin**

In the lecture you discussed the following interaction term for a nucleon $N = (p, n)$, pion $\phi = T^a\phi^a = \mathbf{T}\cdot(\frac{1}{\sqrt{2}}(\pi^+ + \pi^-), \frac{\mathrm{i}}{\sqrt{2}}(\pi^+ - \pi^-), \pi^0)$ interaction

$$\bar{N}\gamma^5 T^a N \phi^a, \tag{14}$$

with $T^a$ the generators of $SU(2)$.

i) Show that the interaction term is invariant under isospin transformations. Recall that the nucleon transforms with the fundamental representation of $SU(2)$ and the components of the pion field with the adjoint representation.
*Hint: Make use of the relation for group generators $T^a$ given in exercise 1 iii).*

ii) Can you think of alternative pion nucleon interaction terms respecting isospin, baryon number, parity and Lorentz invariance?