---

**Exercise 1: Mandelstam variables**

The Mandelstam variables are very useful when calculating $2 \to 2$ scattering processes. They are defined as

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \tag{1}$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \tag{2}$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2, \tag{3}$$

where $p_1$ and $p_2$ are the four-momenta of the incoming and $p_3$ and $p_4$ the four-momenta of the outgoing particles.

i) Show that the following relation holds

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{4}$$

ii) Show that in the case of equal masses $m_1 = m_2 = m_3 = m_4 = m$ the following conditions always hold

$$s \geq 4m^2, \quad t \leq 0, \quad u \leq 0. \tag{5}$$

**Exercise 2: Fermion propagator**

The propagator of spinors is defined as

$$\begin{aligned} S_{F,mn}(x-y) &= \langle 0| T(\psi_m(x)\bar{\psi}_n(y)) |0\rangle \\ &= \Theta(x_0 - y_0) \langle 0| \psi_m(x)\bar{\psi}_n(y) |0\rangle - \Theta(y_0 - x_0) \langle 0| \bar{\psi}_n(y)\psi_m(x) |0\rangle \end{aligned} \tag{6}$$

Comment on the additional minus sign in the time ordering and calculate the fermion propagator in momentum space by inserting the Fourier representation of the field operators, to obtain

$$\tilde{S}_F(p) = \mathrm{i}\frac{\not{p} + m}{p^2 - m^2 + \mathrm{i}\epsilon}. \tag{7}$$

---

**Exercise 3: Muon pair production**

Consider the theory of Quantum Electrodynamics,

$$\mathcal{L}_{QED} = \sum_{f=1}^{2} \bar{\psi}_f \left[ \mathrm{i}\not{D} - m_f \right] \psi_f - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \bar{\psi} \left[ \mathrm{i}\gamma^\mu \left( \partial_\mu + \mathrm{i}e A_\mu \right) - m \right] \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad (8)$$

with $f = 1$ denoting the electron and positron fields with electron mass $m_1 = m_e$ and $f = 2$ the muon and anti-muon fields with muon mass $m_2 = m_\mu$.

Let us have a look at the creation of a muon ($\mu^-$) and an anti-muon ($\mu^+$) from electron ($e^-$) positron ($e^+$) recombination

$$e^+(p_1, r_1) + e^-(p_2, r_2) \to \mu^+(p_1', s_1) + \mu^-(p_2', s_2). \qquad (9)$$

i) Identify all the Feynman diagrams contributing at tree level ($\mathcal{O}(e^2)$) and write down the corresponding matrix elements in momentum space using the Feynman rules familiar from the lecture.

ii) Compute the spin averaged, squared absolute value of the amplitudes in the high energy limit and in the center of mass frame using Feynman gauge ($\xi = 1$).

iii) Use your result from sheet 9 for the differential cross section of a $2 \to 2$ particle process and show that the differential cross section in the case of muon-pair-production is given as

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} \left( 1 + \cos^2 \theta \right), \qquad (10)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant, $\theta$ the scattering angle. Also determine the total cross-section $\sigma$.