

Exercise 1: Wick's theorem

In the lecture you showed Wick's theorem for the time ordered product of two field operators

$$T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right) =: \hat{\phi}(x_1)\hat{\phi}(x_2) : + \langle 0|T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle. \quad (1)$$

Use this result to prove that

$$\begin{aligned} T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\right) &= : \hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3) : + : \hat{\phi}(x_1) \langle 0|T\left(\hat{\phi}(x_2)\hat{\phi}(x_3)\right)|0\rangle \\ &+ : \hat{\phi}(x_2) \langle 0|T\left(\hat{\phi}(x_1)\hat{\phi}(x_3)\right)|0\rangle + : \hat{\phi}(x_3) \langle 0|T\left(\hat{\phi}(x_1)\hat{\phi}(x_2)\right)|0\rangle \end{aligned} \quad (2)$$

Hint: You will need to decompose the fields at some point, as you have done it in the lecture, writing $\hat{\phi} = \hat{\phi}^+ + \hat{\phi}^-$. Also show it for $x_1^0 \geq x_2^0 \geq x_3^0$ at first and discuss afterwards that it holds in all cases.

Exercise 2: Feynman propagator of real scalar field theory

In the lecture you calculated the Feynman propagator of real scalar field theory

$$\Delta_F(x-y) = i \int \frac{e^{-ip(x-y)}}{p^2 - m_0^2 + i\epsilon} \Big|_{\epsilon=0} \frac{d^4p}{(2\pi)^4} \quad (3)$$

i) Prove that the propagator is given as

$$\Delta_F(x-y) = \int (e^{-ip(x-y)}\theta(t_x - t_y) + e^{ip(x-y)}\theta(t_y - t_x)) \frac{d^3p}{(2\pi)^3 2E(\mathbf{p})}, \quad (4)$$

by performing part of the integration.

Hint: Find the poles of the denominator and use the residue theorem. You will have to rescale $\epsilon' = \frac{\epsilon}{2E(\mathbf{p})}$.

ii) We want to show that the Feynman propagator is the Green function of the Klein-Gordon operator

$$(\partial_\mu\partial^\mu + m_0^2)\Delta_F(x-y) = (\partial_\mu\partial^\mu + m_0^2)\langle 0|T\left(\hat{\phi}(x)\hat{\phi}(y)\right)|0\rangle = -i\delta^{(4)}(x-y). \quad (5)$$

Prove that this is indeed the case by letting the operator $(\partial_\mu\partial^\mu + m_0^2)$ act on expectation value of the time ordered product.

Hint: You will need the derivative of the delta distribution, that is only defined in a distributional sense as

$$\left(\frac{\partial}{\partial t_x}\delta(t_x - t_y)\right)\hat{\phi}(x) = -\delta(t_x - t_y)\frac{\partial}{\partial t_x}\hat{\phi}(x). \quad (6)$$

Exercise 3: Mandelstam variables

The Mandelstam variables are very useful when calculating $2 \rightarrow 2$ scattering processes. They are defined as

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad (7)$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \quad (8)$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2, \quad (9)$$

where p_1 and p_2 are the four-momenta of the incoming and p_3 and p_4 the four-momenta of the outgoing particles.

i) Show that the following relation holds

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \quad (10)$$

ii) Show that in the case of equal masses $m_1 = m_2 = m_3 = m_4 = m$ the following conditions always hold

$$s \geq 4m^2, \quad t \leq 0, \quad u \leq 0. \quad (11)$$