

Exercise 1: Quantized complex scalar field

Consider a complex scalar field ϕ

$$S[\phi, \phi^*, \partial_\mu \phi, \partial_\mu \phi^*] = \int ((\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi) d^4x. \quad (1)$$

The general solution of the classical Klein-Gordon equation of a complex scalar field is given as

$$\phi(t, \mathbf{x}) = \int (a(\mathbf{k})e^{-ikx} + b^*(\mathbf{k})e^{ikx}) \frac{d^3k}{(2\pi)^3 2E(k)}, \quad (2)$$

with a and b^* independent complex coefficients.

- i) Calculate the conjugate fields π and π^* and give their explicit form as a function of a and b .
- ii) Let us now quantize the theory, applying the procedure of canonical quantization, identifying

$$\begin{aligned} \phi(x) &\rightarrow \hat{\phi}(x), & \pi(x) &\rightarrow \hat{\pi}(x), \\ \phi^*(x) &\rightarrow \hat{\phi}^\dagger(x), & \pi^*(x) &\rightarrow \hat{\pi}^\dagger(x), \end{aligned} \quad (3)$$

and setting the following commutators unequal zero

$$\begin{aligned} [\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] &= i\delta^3(\mathbf{x} - \mathbf{y}), \\ [\hat{\phi}^\dagger(t, \mathbf{x}), \hat{\pi}^\dagger(t, \mathbf{y})] &= i\delta^3(\mathbf{x} - \mathbf{y}). \end{aligned} \quad (4)$$

All other commutators are zero.

In analogy to the case of a real scalar field, show that the following commutators of the creation- and annihilation operators are non-zero

$$\begin{aligned} [\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{k})] &= 2E(\mathbf{p})(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{k}), \\ [\hat{b}(\mathbf{p}), \hat{b}^\dagger(\mathbf{k})] &= 2E(\mathbf{p})(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{k}). \end{aligned} \quad (5)$$

- iii) On the last sheet you have calculated the Hamilton function of the complex scalar field given as

$$H(\pi, \pi^*, \phi, \phi^*) = \int (\pi^* \pi + (\nabla \cdot \phi^*)(\nabla \cdot \phi) + m^2 \phi^* \phi) d^3x. \quad (6)$$

Apply the procedure of canonical quantization and use the explicit expressions of $\hat{\phi}$ and $\hat{\pi}$ to diagonalize it. Show that it describes two sets of particles with mass m .

iv) On the last sheet you showed that the Lagrangian of the complex scalar field theory is invariant under the following global transformation

$$\phi \rightarrow \phi' = e^{i\alpha} \phi, \quad (7)$$

with associated, conserved Noether current

$$j^\mu = i(\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi). \quad (8)$$

Calculate the charge operator from the zero component of the Noether current

$$\hat{Q} = \int \hat{j}^0 d^3x, \quad (9)$$

then express it as a function of the creation and annihilation operators and calculate the charge of the two kinds of particles, by letting it act on states $\hat{Q}|a(\mathbf{q})\rangle = \hat{Q}\hat{a}^\dagger(\mathbf{q})|0\rangle$ and $\hat{Q}|b(\mathbf{q})\rangle = \hat{Q}\hat{b}^\dagger(\mathbf{q})|0\rangle$.

Comment: In this exercise you showed that the quantized complex scalar field describes two massive charged particles with spin 0.