
Exercise 1: Complex scalar field

Consider the action of a complex scalar field

$$S[\phi, \phi^*, \partial^\mu \phi, \partial^\mu \phi^*] = \int d^4x (\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi). \tag{1}$$

i) Treating ϕ and ϕ^* as dynamical variables, find their conjugate momenta π and π^* and show that the Hamiltonian has the following form

$$H = \int d^3x (\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi). \tag{2}$$

ii) Show hat the Lagrange density is invariant with respect to the following (global) transformation

$$\phi(x) \to \phi'(x) = e^{-\mathrm{i}\alpha} \phi(x), \qquad \phi^*(x) \to \phi^{*\prime} = e^{\mathrm{i}\alpha} \phi^*(x), \tag{3}$$

with α being a constant. Calculate the associated Noether current

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta \phi}{\delta \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \frac{\delta \phi^*}{\delta \alpha} \tag{4}$$

and show that it is conserved

$$\partial_\mu j^\mu = 0. \tag{5}$$

iii) The energy momentum tensor of a theory with $n \in \mathbb{N}$ scalar fields ϕ_n, described by a Lagrange density $\mathcal{L} = \mathcal{L}(\phi_n, \partial_\mu \phi_n)$ is given as

$$\Theta^{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial^\nu \phi_n - g^{\mu\nu} \mathcal{L}. \tag{6}$$

Treating ϕ and ϕ^* as dynamical variables calculate the energy momentum tensor for the theory of a complex scalar field and show that it is symmetric $\Theta^{\mu\nu} = \Theta^{\nu\mu}$.

Exercise 2: The Heisenberg equation

In the lecture you derived the Heisenberg equation of a real and quantized scalar field ϕ

$$\partial^\mu \hat{\phi} = \mathrm{i}[\hat{P}^\mu, \hat{\phi}]. \tag{7}$$

i) Show that in case of $\partial_t \hat{H} = 0$, applying an additional time derivative to the zero component of the Heisenberg equation

$$\partial_t \hat{\phi} = \mathrm{i}[\hat{H}, \hat{\phi}], \tag{8}$$

leads to the Klein-Gordon equation.
Hint: Apply a time derivative and use the Heisenberg equation recursively. Solve the commutator by making use of $[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$.

ii) Show that

$$\hat{\phi}(t, \mathbf{x}) = e^{\mathrm{i}\hat{H}(t-t_0)} \hat{\phi}(t_0, \mathbf{x}) e^{-\mathrm{i}\hat{H}(t-t_0)}, \tag{9}$$

is a solution of the zero component of the Heisenberg equation.

Exercise 3: Unequal time commutator

Consider a real quantized scalar field $\hat{\phi}(x)$ with conjugate field $\hat{\pi}(y)$, where $x^0 \neq y^0$.
Show that the unequal time commutator is given as

$$[\hat{\phi}(x), \hat{\pi}(y)] = \frac{\mathrm{i}}{2} \int \frac{d^3p}{(2\pi)^3} \left(e^{\mathrm{i}p\cdot(x-y)} + e^{-\mathrm{i}p\cdot(x-y)}\right), \tag{10}$$

by using the Fourier transform of $\hat{\phi}$ and $\hat{\pi}$, familiar from the lecture.
Show that this reduces to the equal time commutator, when choosing $x^0 = y^0 = t$.