

Exercise 1: Charge conjugation

We consider a charged particle with mass m and charge q moving in an external electromagnetic field. In the lecture you discussed a relativistic description of such a system, given by the following Dirac equation

$$[i\gamma^\mu (\partial_\mu + iqA_\mu) - m] \psi(x) = 0. \quad (1)$$

Determine the corresponding Dirac equation for the charge conjugated spinor

$$C: \quad \psi \rightarrow \psi^C = i\gamma^2\gamma^0\bar{\psi}^T = i\gamma^2\psi^*, \quad (2)$$

as you have done it in the lecture in case of the free Dirac equation. Comment on the name „charge conjugation“.

Exercise 2: Covariant Electrodynamics

The Lagrange density of Electrodynamics is given as

$$\mathcal{L}(A^\mu(x), \partial^\mu A^\nu(x), t) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu(x) \quad (3)$$

with $x = x^\mu = (t, \mathbf{x})$, $A^\mu(x) = (\phi(x), \mathbf{A}(x))$, where $\phi(x)$ is the scalar- and $\mathbf{A}(x)$ the vector-potential. The current is given as $j^\mu = (\rho(x), \mathbf{j}(x))$, where $\rho(x)$ is the distribution of electric charge and $\mathbf{j}(x)$ the electric current. The electromagnetic tensor is defined as

$$F^{\mu\nu} := \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \quad (4)$$

Use the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \quad (5)$$

to derive the covariant inhomogenous Maxwell equations.

Exercise 3: Left- and right-handed projection operators

We define the following projection operators, projecting on the left- and right-handed components of the Dirac spinors

$$P_L = \frac{\mathbb{1} - \gamma^5}{2} \rightarrow \psi_L := P_L\psi, \quad P_R = \frac{\mathbb{1} + \gamma^5}{2} \rightarrow \psi_R := P_R\psi. \quad (6)$$

i) Show the following properties of projection operators

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_L P_R = P_R P_L = 0. \quad (7)$$

ii) The helicity is defined as the projection of the spin \mathbf{S} of a particle on its momentum \mathbf{p}

$$h(\mathbf{p}) := \frac{\mathbf{S} \cdot \mathbf{p}}{|\mathbf{p}|} = \frac{1}{2|\mathbf{p}|} \begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \end{pmatrix}. \quad (8)$$

Show that in case of the massless Dirac equation, the left- and right-handed spinors defined by $P_L u_s(\mathbf{p})$ and $P_L v_s(\mathbf{p})$ are eigenstates of the helicity operator and determine the eigenvalues. Does this also hold for $m \neq 0$?