WS 19/20
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Quantum Field Theory and the Standard Model of Particle Physics

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Exercise sheet 4

Exercise 1: What does spin 1/2 mean?

In the lecture you discussed another irreducible representation of the Lorentz group, the spinor representation

$$S(\Lambda) = e^{-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}, \qquad \sigma^{\mu\nu} = \frac{\mathrm{i}}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]. \tag{1}$$

Consider a rotation around the z-axis

$$\left(\Lambda^{\mu}{}_{\nu}\right)(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\omega) & -\sin(\omega) & 0 \\ 0 & \sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

It is possible to rewrite the Lorentz transformation of a spinor the following way

$$S(\Lambda) = e^{-\frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}} \quad \rightarrow \quad \omega_{\mu\nu} = \omega \cdot \Omega_{\mu\nu} \;\text{ with }\; \Omega^{\mu}{}_{\nu} = \left.\frac{d\Lambda^{\mu}{}_{\nu}}{d\omega}\right|_{\omega=0} \tag{3}$$

Using the explicit expression for $\sigma^{\mu\nu}$, express the exponential of $S(\Lambda)$ in terms of $\hat{S}^3$, known from the lecture. Calculate the matrix exponential and use your results to compare the behavior of a spinor $\psi' = S(\Lambda)\psi$, to that of a vector $A^{\mu\prime} = \Lambda^{\mu}{}_{\nu}A^{\nu}$ when rotated by $\omega = 2\pi$.

Exercise 2: Bilinear covariants I

i) Use the definition of the gamma matrices from the lecture, to show that

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}. \tag{4}$$

Knowing that $\gamma_0^{\dagger} = \gamma_0$ and $\gamma_i^{\dagger} = -\gamma_i$, use your previous results to show that

$$\gamma^{\mu\dagger} = \gamma^0\gamma^{\mu}\gamma^0. \tag{5}$$

ii) In the lecture you found the following Lorentz transformation of a Dirac spinor ψ

$$\psi(x) \rightarrow \psi'(x') = S(\Lambda)\psi(x), \qquad S(\Lambda) = e^{-\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}, \tag{6}$$

with $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_{\mu}, \gamma_{\nu}]$.
Furthermore you have shown that $\bar{\psi}\psi$ is a Lorentz scalar, therefore the inverse transformation is given by

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x')S^{-1}(\Lambda), \qquad S^{-1}(\Lambda) = e^{\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}}. \tag{7}$$

Show that $S^{-1}(\Lambda)$ can also be written as

$$S^{-1}(\Lambda) = (\gamma^0 S(\Lambda)\gamma^0)^{\dagger} \tag{8}$$

(instead of $S^{-1}(\Lambda) = S^{\dagger}(\Lambda)$ as one would expect naively).

Hint: Use the result of i)

Exercise 3: Bilinear covariants II

Recall the representation of an infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu} \tag{9}$$

and the Lorentz transformation of a spinor

$$S(\Lambda) = 1 - \frac{\mathrm{i}}{4}\sigma_{\mu\nu}\delta\omega^{\mu\nu} + o(\delta\omega^2), \qquad \sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]. \tag{10}$$

i) Using these transformations, show that the equation

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^\nu, \tag{11}$$

familiar from the lecture, is satisfied up to first order in $\delta\omega$.

ii) In the lecture you showed that $\bar{\psi}\psi$ transforms as a Lorentz scalar and $\bar{\psi}\gamma^\mu\psi$ transforms as a Lorentz vector. Furthermore you introduced an additional gamma matrix $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ with the transformation property

$$S^{-1}(\Lambda)\gamma^5 S(\Lambda) = \det(\Lambda)\gamma^5. \tag{12}$$

Prove the behavior under Lorentz transformation of the following bilinear covariants

$$\bar{\psi}\gamma^5\psi \to \det(\Lambda)\bar{\psi}\gamma^5\psi \qquad \text{pseudoscalar} \tag{13}$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \to \det(\Lambda)\Lambda^{\mu}{}_{\nu}\bar{\psi}\gamma^\nu\gamma^5\psi \qquad \text{axial vector} \tag{14}$$

$$\bar{\psi}\sigma^{\mu\nu}\psi \to \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\bar{\psi}\sigma^{\alpha\beta}\psi \qquad \text{antisymmetric tensor} \tag{15}$$

Hint: Equation (11) will be very useful.