

**Exercise 1: Dirac and Klein-Gordon equation**

Let  $\psi$  be a Dirac spinor solving the Dirac equation

$$[i\gamma^\mu\partial_\mu - m]\psi(x) = 0. \quad (1)$$

Show that it is also a solution of the Klein-Gordon equation

$$[\partial_\mu\partial^\mu + m^2]\psi(x) = 0. \quad (2)$$

**Exercise 2: Continuity equation in relativistic quantum mechanics II**

Let the Dirac spinor  $\psi$  be a solution of the Dirac equation

$$[i\gamma^\mu\partial_\mu - m]\psi(x) = 0. \quad (3)$$

Show that the current

$$j^\mu(x) = \bar{\psi}\gamma^\mu\psi \quad (4)$$

satisfies a continuity equation

$$\partial_\mu j^\mu(x) = 0. \quad (5)$$

Discuss whether  $j^0 = \rho$  is positive definite in the case of the Dirac equation.

**Exercise 3: Permutation group**

Consider a set of three distinguishable elements  $(a, b, c)$ . There are several possibilities to permute the elements  $((a, b, c) \rightarrow (b, a, c) \dots)$ .

- i) Construct all possible  $3 \times 3$  matrices  $s_i$  that permute the elements of the set, in the sense

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}. \quad (6)$$

- ii) Motivate, that the set  $S_3$  of all matrices  $s_i$  forms a group.  
iii) Show that the group is non-Abelian. Is there an Abelian subgroup?

**Exercise 4: The Lorentz group**

The set of all real  $4 \times 4$  matrices  $\Lambda \in \mathbb{R}^{4 \times 4}$ , that preserve the Minkowski metric  $g = \text{diag}(1, -1, -1, -1)$  via the following relation

$$\Lambda^T g \Lambda = g, \quad (7)$$

is called Lorentz group  $O(1, 3) = \{\Lambda \in \mathbb{R}^{4 \times 4} | \Lambda^T g \Lambda = g\}$ .

- i) Discuss that this is indeed a group with respect to the group axioms.
- ii) The Lorentz group is a Lie group and an irreducible (vector) representation of the group is given as

$$\Lambda = e^{i\theta_{\mu\nu} V^{\mu\nu}}, \quad V^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}, \quad (8)$$

where  $K_i$  and  $J_i$  are the  $4 \times 4$  generating matrices of rotations and boosts satisfying the Lie algebra of the group

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k. \quad (9)$$

Given that

$$J_1 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad K_3 = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad (10)$$

construct the missing  $J_3$ ,  $K_1$  and  $K_2$ .

- iii) Show that

$$\Lambda(\theta_z) = e^{i\theta_z V^{12}}, \quad \Lambda(\phi_z) = e^{i\phi_z V^{03}}, \quad (11)$$

leads to the familiar rotation matrix around the z-axis and the boost in z-direction with rapidity  $\phi = \text{artanh}(v)$ .