

\*\*\*\*\*Bonus sheet\*\*\*\*\*

### Exercise 1: QCD running coupling

In the lecture you discussed that  $\alpha_s = \frac{g^2}{4\pi}$ , with  $g$  the strong coupling of QCD, is a function of energy. This running coupling is given by the following differential equation,

$$Q^2 \frac{d\alpha_s}{dQ^2} = -\beta_0 \alpha_s^2(Q^2) + \mathcal{O}(\alpha_s^3), \quad (1)$$

where  $Q^2$  is the energy scale of the system and  $\beta_0$  is given as

$$\beta_0 = \frac{11N_c - 2N_f}{12\pi}, \quad (2)$$

with  $N_c$  colors and  $N_f$  flavors.

- i) Solve the differential equation up to first order, ignoring all terms  $\mathcal{O}(\alpha_s^3)$  and higher.
- ii) The value for  $\alpha_s$  at the energy scale of the mass of the Z-boson,  $Q = M_Z = 91.1$  GeV is

$$\alpha_s(Q^2 = M_Z^2) = 0.12. \quad (3)$$

Use  $Q = 10$  GeV with  $N_f = 5$  flavors and calculate the value of  $\alpha_s$ .

### Exercise 2: Fermion mass generation

Consider the following theory for real scalar fields and fermions

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \bar{\psi} i \gamma^\mu \partial_\mu \psi - g \phi \bar{\psi} \psi, \quad (4)$$

with the following potential

$$V(\phi) = \frac{\lambda}{4!} (\phi^2 - v^2)^2. \quad (5)$$

- i) Sketch the potential  $V(\phi)$  of the scalar field. What is the vacuum expectation value  $\langle \phi \rangle = \langle 0 | \phi | 0 \rangle$ ?
- ii) Reparametrise the scalar field by its fluctuations around the expectation value  $\phi = \langle \phi \rangle + \chi$  and express  $\mathcal{L}$  in terms of the field variables  $\psi$  and  $\chi$ .
- iii) Identify the particle masses  $m_\psi$  and  $m_\chi$  and the basic vertices for all interaction terms in this theory, without specifying the Feynman rules. Is there a Goldstone boson? Discuss without calculations what changes in case of a complex scalar field.

**Exercise 3: Electroweak theory**

Consider the following terms of the electroweak Lagrangian for the first generation of leptons

$$\mathcal{L}_f = \bar{\ell}_L i\gamma^\mu \tilde{D}_\mu \ell_L + \bar{e}_R i\gamma^\mu D_\mu e_R, \quad (6)$$

with  $\ell_L = (\nu_L, e_L)$  the left-handed  $SU(2)$  isospin doublet and  $e_R$  the right-handed singlet. The covariant derivatives are given as

$$(\tilde{D}_\mu)_{bc} = \partial_\mu \delta_{bc} + \frac{ig'}{2} B_\mu \delta_{bc} - ig T_{bc}^a W_\mu^a, \quad (7)$$

$$D_\mu = \partial_\mu + ig' B_\mu, \quad (8)$$

where  $T^a$  are the generators of  $SU(2)$ .

i) Insert the covariant derivatives and simplify as much as possible to identify the kinetic and interaction terms.

ii) Identify the currents

$$j_{\nu_L}^\mu = \bar{\nu}_L \gamma^\mu \nu_L, \quad j_{e_L}^\mu = \bar{e}_L \gamma^\mu e_L, \quad j_{e_R}^\mu = \bar{e}_R \gamma^\mu e_R, \quad (9)$$

to rewrite the interaction terms and replace the coupling  $g'$  using the Weinberg mixing angle

$$\tan \theta_W = \frac{g'}{g}. \quad (10)$$

iii) The gauge fields  $B_\mu$  and  $W_\mu$  are connected to the physical gauge bosons  $\{W^\pm, Z, A\}$ . Use the following redefinitions and rotations to rewrite the Lagrangian in terms of physical fields

$$W_\mu^+ = \frac{1}{\sqrt{2}} (W_\mu^1 - iW_\mu^2), \quad (11)$$

$$W_\mu^- = \frac{1}{\sqrt{2}} (W_\mu^1 + iW_\mu^2), \quad (12)$$

$$B_\mu = \cos \theta_W A_\mu - \sin \theta_W Z_\mu, \quad (13)$$

$$W_\mu^3 = \cos \theta_W Z_\mu + \sin \theta_W A_\mu. \quad (14)$$

Identify the electrical charge  $e$  as a function of  $\theta_W$  and  $g$ .