
Exercise 1: Isospin

In the lecture you discussed the following interaction term for a nucleon $N = (p, n)$, pion $\phi = T^a\phi^a = \mathbf{T} \cdot (\frac{1}{\sqrt{2}}(\pi^+ + \pi^-), \frac{\mathrm{i}}{\sqrt{2}}(\pi^+ - \pi^-), \pi^0)$ interaction

$$\bar{N}\gamma^5 T^a N \phi^a, \tag{1}$$

with T^a the generators of $SU(2)$.

i) Show that the interaction term is invariant under isospin transformations. Recall that the nucleon transforms with the fundamental representation of $SU(2)$ and the components of the pion field with the adjoint representation.
Hint: Use the following relation, valid for $SU(N)$

$$T^a_{li}T^a_{no} = \frac{1}{2}\left(\delta_{lo}\delta_{in} - \frac{1}{N}\delta_{li}\delta_{no}\right). \tag{2}$$

ii) Can you think of alternative pion nucleon interaction terms respecting isospin, baryon number, parity and Lorentz invariance?

Exercise 2: Adjoint Representation

On the last exercise sheet you discussed the adjoint representation of $SU(N)$. Given a $N \times N$ matrix $U \in SU(N)$ in fundamental representation, the adjoint representation is given as

$$D^{ab}(U) = 2\mathrm{tr}(U^\dagger T^a U T^b), \quad a, b \in \{1, ..., N^2 - 1\}. \tag{3}$$

Using the relation for the generators T^a from exercise 1, prove that for two matrices $U, U' \in SU(N)$ in fundamental representation, one has

$$D^{ab}(U)D^{bc}(U') = D^{ac}(UU'). \tag{4}$$

Exercise 3: Yang-Mills theory

Consider the Yang-Mills Lagrangian of an $SU(N)$ gauge field $A^\mu = A^{\mu a}T^a$

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu}\right), \tag{5}$$

with T^a the generators of $SU(N)$, normalized to $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.

i) Given the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu], \tag{6}$$

express the Lagrangian in terms of $A^{\mu a}$ and identify the self interaction terms.

ii) Show that adding a mass term

$$\mathcal{L}_m = m^2\mathrm{tr}(A^\mu A_\mu), \tag{7}$$

leads to a Lagrangian, that is not gauge invariant.