

Exercise 1: Bhabha-scattering

Consider the theory of Quantum Electrodynamics for electrons, positrons and photons,

$$\mathcal{L}_{QED} = \bar{\psi} [i\not{D} - m] \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \bar{\psi} [i\gamma^\mu (\partial_\mu + ieA_\mu) - m] \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \quad (1)$$

Electron (e^-) - Positron (e^+) scattering is referred to as Bhabha-scattering

$$e^+(p_1, r_1) + e^-(p_2, r_2) \rightarrow e^+(p'_1, s_1) + e^-(p'_2, s_2). \quad (2)$$

- i) Identify all the Feynman diagrams contributing at tree level ($\mathcal{O}(e^2)$) and write down the corresponding matrix elements in momentum space using the Feynman rules familiar from the lecture.
- ii) Compute the spin averaged squared absolute value of the amplitudes (keep in mind that there could be interference terms!) in the high energy limit and in the center of mass frame using Feynman gauge ($\xi = 1$).
- iii) Recall from sheet 9 that the differential cross section of a $2 \rightarrow 2$ particle process was given as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |M_{fi}|^2, \quad (3)$$

in the case of equal masses for incoming and outgoing particles. Use your previous result to show that the differential cross section in the case of Bhabha-scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2} \left(\frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{1 + \cos^2(\theta)}{2} - 2 \frac{\cos^4(\theta/2)}{\sin^2(\theta/2)} \right), \quad (4)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant, θ the scattering angle and E is the respective energy of electron and positron in the center-of-mass system.

Hint: Use Mandelstam variables.

Exercise 2: $SU(N)$ gauge theory

In the lecture you derived the covariant derivative of $SU(N)$ gauge theory

$$D_\mu = \partial_\mu - igA_\mu^a T^a, \quad (5)$$

with $SU(N)$ gauge field A_μ^a and the generators T^a of the $SU(N)$ group.

Use the covariant derivative to calculate the field strength tensor $F_{\mu\nu}^a$ of $SU(N)$ gauge theory via

$$F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu]. \quad (6)$$

Exercise 3: Adjoint Representation

Consider the Lie group $SU(N)$. In the lecture you derived the fundamental representation of the group given by the $N \times N$ matrices $U \in SU(N)$

$$U = e^{-i\theta^a T^a}, \quad a = 1, \dots, N^2 - 1, \quad (7)$$

with generators T^a of the Lie group. Let us consider normalized generators that satisfy the Lie algebra

$$[T^a, T^b] = if^{abc}T^c, \quad \text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}, \quad (8)$$

with antisymmetric structure constants f^{abc} . A field ϕ^a is said to transform with the adjoint representation of the group $SU(N)$ if the associated matrix valued field, defined by $\phi := \phi^a T^a$ transforms as

$$\phi' = U\phi U^\dagger. \quad (9)$$

- i) Show that the transformation of the components ϕ^a , and therefore the adjoint representation $D^{ab}(U)$ is given as

$$\phi^a \rightarrow \phi'^a = D^{ab}(U)\phi^b = 2\text{tr}(U^\dagger T^a U T^b)\phi^b. \quad (10)$$

- ii) Letting the covariant derivative act on the matrix valued field $\phi = \phi^a T^a$ reads

$$D_\mu \phi(x) = \partial_\mu \phi(x) - ig[A_\mu, \phi(x)]. \quad (11)$$

Show that this transforms as

$$(D_\mu \phi)'(x) = U(x)(D_\mu \phi(x))U^\dagger(x), \quad (12)$$

with $U(x) \in SU(N)$ and that

$$(D_\mu \phi)^a = \partial_\mu \phi^a - gf^{abc}A_\mu^c \phi^b. \quad (13)$$