Exercise Sheet 11 **Theoretische Physik 4: Quantenmechanik**

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Exercise 1 (8 points)

For the simple harmonic oscillator, with frequency ω, we have had a dedicated chapter solving it in the Schrödinger picture. Now let's see what happens in the Heisenberg picture.

1. (1/8) For any potential, what is the definition of the Heisenberg picture operators $\hat{x}_H(t)$ and $\hat{p}_H(t)$? What are their eigenvectors $|x,t\rangle_H$ and $|p,t\rangle_H$? Do these eigenvectors form a complete set?
2. (3/8) Derive the equations of motion of $\hat{x}_H(t)$ and $\hat{p}_H(t)$ and solve them.
3. (1/8) Find the position representation of $\hat{x}_H(t)$ and $\hat{p}_H(t)$.
4. (2/8) Calculate the commutators $[\hat{x}_H(t_1), \hat{x}_H(t_2)]$, $[\hat{p}_H(t_1), \hat{p}_H(t_2)]$, and $[\hat{x}_H(t_1), \hat{p}_H(t_2)]$. What is the key difference with the Schrödinger picture operators?
5. (1/8) Show that $[\Delta x(t_1)]^2[\Delta x(t_2)]^2 \geq \dfrac{\hbar^2}{4m^2\omega^2}\sin^2[\omega(t_2-t_1)]$

Exercise 2 (12 points)

Consider a classical vector $\mathbf{v}=\sum_\mu v_\mu \mathbf{e}_\mu$, with components v_μ in a right-handed cartesian co-ordinate basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

1. (1/12) Show that a rotation of angle α around the cartesian $+z$-axis changes the vector $\mathbf{v}$ to a new vector $\mathbf{v}'$ given by

$$v'_\mu = R^{(z)}_{\mu\nu}(\alpha) v_\nu, \quad \text{with matrix representation} \quad \mathbf{R}^{(z)}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. (2/12) Show that an infinitesimal small rotation angle δa gives an infinitesimal change $\delta\mathbf{v} = \mathbf{v}' - \mathbf{v} = \delta\alpha(\mathbf{e}_z \times \mathbf{v})$
3. (1/12) Show that an infinitesimal small rotation angle $\delta\phi$ around any unit vector $\mathbf{n}$, gives an infinitesimal change $\delta\mathbf{v} = \mathbf{v}' - \mathbf{v} = \delta\phi(\mathbf{n} \times \mathbf{v})$

Given a rotation $\mathbf{R}$ we can consider the operator $\hat{D}_\mathbf{R}$ which would rotate a state localized at position $|\mathbf{r}\rangle$ to a state localized at the rotated position $|\mathbf{r}'\rangle = \hat{D}_\mathbf{R}|\mathbf{r}\rangle$. Since rotations should preserve the norm of the states, it must be that $\hat{D}_\mathbf{R}$ is a unitary operator. Given a general system state $|\psi\rangle$, a rotation takes us from $\psi(\mathbf{r}) = \langle\mathbf{r}|\psi\rangle$ to $\psi(\mathbf{r}') = \langle\mathbf{r}'|\psi\rangle$.

4. (3/12) Show that for an infinitesimal small rotation angle $\delta\phi$ around unit vector $\mathbf{n}$ the wavefunction changes as $\psi(\mathbf{r}') = (1 + \delta\phi\mathbf{n}\cdot(\mathbf{r}\times\boldsymbol{\nabla}))\psi(\mathbf{r})$, and then show that the corresponding infinitesimal rotation operator is $\hat{D}_{\mathbf{R}^{(n)}(\delta\phi)} = \hat{I} - \frac{i}{\hbar}\delta\phi\,\mathbf{n}\cdot\hat{\mathbf{L}}$

One can chain together many infinitesimal small rotation $\delta\phi$ around a single unit vector $\mathbf{n}$, to a finite ϕ rotation around $\mathbf{n}$. We can imagine this construction by breaking up the angle ϕ into many small angles $\delta\phi = \phi/N$ and making the subdivisions N arbitraraly large (infinity many).

5. (2/12) Show that the rotation of angle ϕ around unit vector $\mathbf{n}$ results in the rotation operator is $\hat{D}_{\mathbf{R}^{(n)}(\phi)} = \mathrm{e}^{-\frac{i}{\hbar}\phi\mathbf{n}\cdot\hat{\mathbf{L}}}$.

Now let's think about it in reverse. We make a demand:

Under a rotation of angle ϕ around unit vector $\mathbf{n}$,
we demand that kets transform as $|\psi_R\rangle = \mathrm{e}^{-\frac{i}{\hbar}\phi\mathbf{n}\cdot\hat{\mathbf{J}}}|\psi\rangle$,
where $\hat{\mathbf{J}}$ a vector operator (an operator whose
expectation values behave like classical vectors).

6. (3/12) Show that it must be the case that $[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$. Given this, discuss what these operators $\hat{\mathbf{J}}$ could be.