Exercise Sheet 10 **Theoretische Physik 4: Quantenmechanik**

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Exercise 1 (4 points)

The Hamiltonian of a system, living in the 3-dimensional Hilbert space $\mathcal{H}^{(3)}$, is given by

$$\hat{H} = \varepsilon \left(|1\rangle\langle 1| - 2|2\rangle\langle 2| - 3|3\rangle\langle 3| \right),$$

where $\{|n\rangle\}$ is an orthonormal basis of $\mathcal{H}^{(3)}$. At time $t = 0$ the system is in the state

$$|\psi(0)\rangle = N(\sqrt{3}\,|1\rangle - i\,|2\rangle + i\,|3\rangle).$$

1. (1/4) Find the 3×3 matrix representation of $\hat{H}$ in the $\{|n\rangle\}$, and find the eigenvalues and eigenvectors.
2. (2/4) At $t = 0$, if we measure the energy of the system, what are the possible outcomes and with what probability? If we prepare numerous systems like this, and measure the energy once on each, what is the expectation value $\langle \hat{H}(0) \rangle$?
3. (1/4) Find the state $|\psi(t)\rangle$ at an arbitrary later time $t > 0$ and determine the expectation value $\langle \hat{H}(t) \rangle$.

Exercise 2 (5 points)

A three-state system, living in the 3-dimensional Hilbert space $\mathcal{H}^{(3)}$, has energy eigenvalues $E_1 = \hbar\omega, E_2 = 0, E_3 = -\hbar\omega$ with corresponding energy eigenvector $|1\rangle, |2\rangle, |3\rangle$. In the eigenbasis of the Hamiltonian, another observable $\hat{A}$ is

$$\hat{A} = \frac{\alpha}{\sqrt{2}}(|1\rangle\langle 2| + |2\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|),$$

where the constant $\alpha > 0$ carries the units of the observable.

1. (2/5) Find the 3×3 matrix representation of $\hat{A}$ in the eigenbasis of the Hamiltonian.
2. (1/5) What are the possible measurement outcomes of $\hat{A}$?
3. (2/5) At $t = 0$ you measure $\hat{A}$ and obtain α. What is the probability that a subsequent measurement of A at time t will again yield α?

Exercise 3 (5 points)

A system, living in the 4-dimensional Hilbert space $\mathcal{H}^{(4)}$, has unperturbed Hamiltonian $\hat{H}_o$ and a perturbation $\hat{V}$, given in the eigenbasis $\{|n^{(o)}\rangle\}$ of $\hat{H}_o$ by the representations

$$\mathbb{H}_o = E_o \begin{pmatrix} 15 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \qquad \mathbb{V} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. (1/5) Find the eigenvalues and eigenstates of $\hat{H}_o$.
2. (2/5) Find the exact eigenvalues of the full Hamiltonian $\hat{H} = \hat{H}_o + \lambda\hat{V}$.

3. (2/5) Now treat $\hat{H} = \hat{H}_o + \lambda \hat{V}$ as a perturbation problem, where $\hat{H}_o$ is the unperturbed problem, and $\lambda \ll E_o$. Compute the first-order perturbative energies. Compare and discuss with the exact results.

Exercise 4 (6 points)

A particle of mass m moves in a two–dimensional isotropic harmonic potential,

$$\hat{H}_0 = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2).$$

1. (2/6) Write down the energies and eigenfunctions of the three lowest levels.
2. (4/6) The potential is modified to

$$\hat{H} = \hat{H}_0 + \lambda \hat{x}\hat{y} \quad \lambda > 0.$$

   Compute the first-order correction to the energies of the three lowest levels.