Exercise Sheet 9 **Theoretische Physik 4: Quantenmechanik**

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Exercise 1 (12 points)

Consider a complete orthonormal basis $\{|e_n\rangle\}$ of the d-dimensional Hilbert space $\mathcal{H}^{(d)}$ where our system lives, where orthonormality in bra-ket notation is $\langle e_n|e_m\rangle = \delta_{nm}$.

1. (1/12) Show that the operator $\hat{I} = \sum_{n=1}^{d} |e_n\rangle\langle e_n|$ is simply the identity operator (this is also called resolution of identity or completeness relation).

Now lest define another complete orthonormal basis $\{|\tilde{e}_n\rangle\}$ of $\mathcal{H}^{(d)}$, generally different to the first basis (meaning $\langle e_n|e_m\rangle = \delta_{nm}$ and $\langle \tilde{e}_n|\tilde{e}_m\rangle = \delta_{nm}$, but $\langle \tilde{e}_n|e_m\rangle \neq \delta_{nm}$).

2. (2/12) Show that the operator $\hat{U} = \sum_{n=1}^{d} |\tilde{e}_n\rangle\langle e_n|$ is a unitary operator. Write the matrix $\mathbf{U}$, the operator representation of $\hat{U}$ in $\{|e_n\rangle\}$, and use index algebra to show the same thing. What role does $\hat{U}$ and $\hat{U}^\dagger$ appear to be playing (consider what $\hat{U}\,|e_n\rangle$ and $\hat{U}^\dagger\,|\tilde{e}_n\rangle$ give you)?

Suppose an observable, some hermitian operator $\hat{O}$ acting on $\mathcal{H}^{(d)}$, is represented in $\{|e_n\rangle\}$ and $\{|\tilde{e}_n\rangle\}$ by the matrices $\mathbf{O}$ and $\tilde{\mathbf{O}}$ respectively.

3. (2/12) Show that $\mathbf{U}^\dagger \mathbf{O} \mathbf{U} = \tilde{\mathbf{O}}$. What role does the U appear to play here? Is it compatible with your previous conclusion?
4. (1/12) Show that $\mathrm{Tr}(\mathbf{O}) = \mathrm{Tr}(\tilde{\mathbf{O}})$, making the trace an basis independent invariant.

An operator $\hat{P}$ acting on $\mathcal{H}^{(d)}$ is called a projector operator if $\hat{P}^\dagger = \hat{P}$ and $(\hat{P})^2 = \hat{P}$.

5. (2/12) Is $P_n = |e_n\rangle\langle e_n|$ a projector operator? How about $|e_n\rangle\langle e_n| + |e_m\rangle\langle e_m|$? How about $\alpha|e_n\rangle\langle e_n|$, where $\alpha \in \mathbb{C}$?
6. (2/12) Consider any $|\psi\rangle, |\phi\rangle \in \mathcal{H}^{(d)}$ such that $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$ and $\langle\psi|\phi\rangle \neq 0$. Show that, for any integer $n \geq 1$, $(|\psi\rangle\langle\psi|)^n = |\psi\rangle\langle\psi|$ and $(|\psi\rangle\langle\phi|)^n = \langle\phi|\psi\rangle^{n-1}\,|\psi\rangle\langle\phi|$. Are any of these projector operators?
7. (2/12) Consider two projector operators $\hat{P}$ and $\hat{Q}$ acting on $\mathcal{H}^{(d)}$, and show that:
   - i) For the sum $\hat{P} + \hat{Q}$ to be projector it must be that $\hat{P}\hat{Q} = 0$
   - ii) For the product $\hat{P}\hat{Q}$ to be a projector it must be that $[\hat{P}, \hat{Q}] = 0$.

Exercise 2 (8 points)

Consider a system spin $s = 1/2$, in the presence of a magnetic field $B_o > 0$ along the z-axis, $\hat{H}_o = -2B_o\hat{S}^z$. Lets consider what happens if we put this system into a thermal bath, at temperature T.

1. (1/8) Find the operator $\hat{H}_o = \sum_{n,m} h_{nm}|e_n\rangle\langle e_m|$ in the eigenbasis $\{|e_n\rangle\}$ of $\hat{S}^z$, and find the eigenvalues and eigenstates.
2. (2/8) Find the thermal density matrix $\hat{\rho} = \mathrm{e}^{-\beta\hat{H}_o}/Z$ in the eigenbasis of $\hat{S}^z$, where Z the partition function, and discuss the limits $T \to 0^+$ and $T \to +\infty$. How does the density matrix look like in the limits and why? Is there a pure state for some T?

3. (1/8) Find the magnetization $M^z(T) = \langle \hat{S}^z \rangle$ in the thermal state. Sketch the curve $M^z(T)$ and discuss the limits $T \to 0^+$ and $T \to +\infty$.

Suppose the system is left for a long enough time that it equilibrates. Then suddenly, at time $t = 0$, you apply an additional field $B > 0$ along the x-axis, such that your new hamiltonian is $\hat{H} = \hat{H}_o - 2B\hat{S}^x$

4. (3/8) What is the density matrix $\hat{\rho}(t = 0)$? Find the time evolved density matrix $\hat{\rho}(t) = \mathrm{e}^{-i\hat{H}t/\hbar}\hat{\rho}(t = 0)\mathrm{e}^{i\hat{H}t/\hbar}$ in the eigenbasis of $\hat{S}^z$.

5. (1/8) Find $M^z(T, t)$ and $M^x(T, t)$ for $t > 0$. Discuss their behavior as a function of both t and T.