Exercise Sheet 8 **Theoretische Physik 4: Quantenmechanik**

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Exercise 1 (8 points)

1. (2/8) Consider a particle with spin $s = \hbar/2$. What is the most general physical wavefunction you can down in spin-space?

2. (2/8) We carry out measurements in the lab on the above system. We find the expectation values $\langle \hat{S}_x \rangle = \hbar c_x$, $\langle \hat{S}_y \rangle = \hbar c_y$, $\langle \hat{S}_z \rangle = \hbar c_z$. Use this information to determine the wavefunction fully.

3. (2/8) In the previous question, was there too much information given by the measurements? What is the minimum amount of information needed to make it work? If instead we had measured expectation values of commuting operators would this determination be possible or not and why?

4. (2/8) After measurements lets say we found that $\Psi = \frac{1}{\sqrt{2}}\Psi_{+z} + \mathrm{e}^{i\pi/4}\frac{1}{\sqrt{2}}\Psi_{-z}$, where $\Psi_{\pm z}$ the eigenstates of $\hat{S}_z$. What would be the spin projection needed to measure, so that the uncertainty is minimized?

Exercise 2 (12 points)

In this problem we will examine the difference that arises when one measures the spin of a particle, or equivalently the state of a particle, in a sequence of Stern–Gerlach experiments. Assume we send a beam of particles with spin $s = \hbar/2$ through a Stern–Gerlach apparatus whose magnetic field is oriented along the z-axis, i.e. $\mathbf{B} \parallel \mathbf{e}_z$. The inhomogeneity of the field splits the beam into two, one deflected in the $+z$ direction and the other in the $-z$ direction. We denote the state of the particles:

$$\Psi_{+z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for particles deflected in } +\mathbf{e}_z,\ \Psi_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for particles deflected in } -\mathbf{e}_z.$$

Measuring the spin along the unit vector $\mathbf{n}(\phi) = \sin\phi\,\mathbf{e}_x + \cos\phi\,\mathbf{e}_z$ in the xz-plane means measuring the rotated operator $\hat{S}_n(\phi) = e^{-\frac{i\phi}{\hbar}\hat{S}_y}\ \hat{S}_z\ e^{\frac{i\phi}{\hbar}\hat{S}_y}$.

1. (3/12) Show that

$$\hat{S}_n(\phi) = \frac{\hbar}{2}\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}.$$

2. (3/12) Compute the expectation values $\langle \hat{S}_x \rangle$, $\langle \hat{S}_y \rangle$, $\langle \hat{S}_z \rangle$ when the system is in the Ψ state:

$$\text{i) } \Psi = \Psi_{+z}, \quad \text{ii) } \Psi = \Psi_{-z}, \quad \text{iii) } \Psi = \tfrac{1}{\sqrt{2}}(\Psi_{+z} + \Psi_{-z}), \quad \text{iv) } \Psi = \tfrac{1}{\sqrt{2}}(\Psi_{+z} - \Psi_{-z}).$$

Which spin directions do the states in (iii) and (iv) represent?

3. (3/12) Define a new state $\Phi_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, with $\theta \in [0, 2\pi)$, and compute the expectation value $\langle \hat{S}_n(\phi) \rangle$ for any θ, ϕ. Use this to find the eigenstates $\Psi_{+\hat{n}}$ and $\Psi_{-\hat{n}}$ corresponding to spin along $+\mathbf{n}(\phi)$ and $-\mathbf{n}(\phi)$, respectively

4. (3/12) Consider a sequence of two Stern–Gerlach measurements, as illustrated in the figure. Prepare the particle in the state Ψ_{+z}, send it through a Stern–Gerlach apparatus with field along $+\mathbf{n}(\phi)$, and then send each emerging beam into a second Stern–Gerlach apparatus with field along $+\mathbf{e}_z$. Compute, as functions of ϕ, the probabilities that the particle is found in each of the four outputs (labelled (1), (2), (3), (4) in the figure). Discuss the special cases $\mathbf{n} = \mathbf{e}_z$ and $\mathbf{n} = \mathbf{e}_x$.