Exercise Sheet 7 — **Theoretische Physik 4: Quantenmechanik**

SoSe 2025 — Prof. Dr. C. Gros

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Exercise 1 (6 points)

In the lectures, the spherical Bessel and Neumann functions were defined as

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\sin x}{x}\right), \; n_l(x) = -(-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{\cos x}{x}\right).$$

Their asymptotic behaviours for small and large argument x is often used in the central potentials to find limiting cases.

1. (3/6) Using the definition above, show that for $x \ll 1$ we have the approximate forms $j_l(x) \simeq \frac{x^l}{(2l+1)!!}$, and $n_l(x) \simeq -\frac{(2l-1)!!}{x^{l+1}}$.
2. (3/6) Using the definition above, show that for $x \gg 1$ we have the asymptotic behavior $j_l(x) \sim \frac{\sin\left(x - \frac{l\pi}{2}\right)}{x}$, and $n_l(x) \sim -\frac{\cos\left(x - \frac{l\pi}{2}\right)}{x}$.

Exercise 2 (5 points)

1. (3/5) In the scattering problem for real spherically symmetric potentials $V(r)$ we saw how the radial wavefunction will include a phase-shift $\delta_l(k)$. Argue on general grounds, regardless of potential (assuming only that it is short range enough) why $\delta_l(k)$ must be an odd function of k, and show that $S_l(-k) = S_l^*(k)$.
2. (2/5) Check explicitly that the above holds for the radial square well $V(\mathbf{r}) = \begin{cases} 0 & r \geq a \\ -V_o & r < a \end{cases}$, where $V_o > 0$.

Exercise 3 (9 points)

Consider the hard sphere potential

$$V(r) = \begin{cases} +\infty, & 0 \leq r < a, \\ 0, & r > a. \end{cases}$$

1. (1/9) Can this potential have bound states? Explain why.
2. (2/9) Considering a scattering experiment off this potential, find the phase-shift $\delta_l(k)$.
3. (3/9) Find the phase-shift in the short and long wavelength limits, and discuss any bounds placed on l in these two limits.
4. (3/9) Find the cross-section in the short and long wavelength limits. What should be the classical cross section? Compare to the quantum and briefly discuss. You can use the closed form of the integer sums:

$$\sum_{n=0}^{N} n = \frac{N(N+1)}{2}, \quad \sum_{n \in \text{even}}^{N} n = \frac{N(N+2)}{4}, \quad \sum_{n \in \text{odd}}^{N} n = \frac{N^2}{4}$$