Exercise Sheet 6 **Theoretische Physik 4: Quantenmechanik**

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Exercise 1 (10 points)

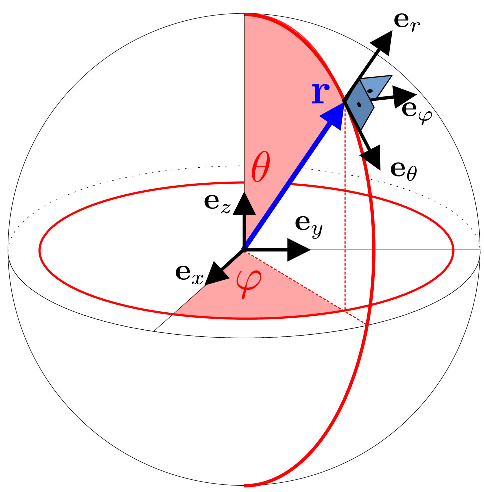

FIG. 1. Spherical coordinates (r, φ, θ), and the spherical unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$.

In the figure you will find the definitions of spherical coordinates (r, φ, θ). Just like the cartesian $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ unit vectors, spherical unit vector $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$, are orthonormal (normalized to unity and orthogonal between them). The only difference is, while cartesian unit vectors are fixed across all space, spherical unit vector orientation depends on position in space. In the figure we show an example of spherical unit vector for one $\mathbf{r}$ position. The relation of cartesian to spherical coordinates is given by

$$\begin{aligned} x(r, \varphi, \theta) &= r \cos\varphi \sin\theta \\ y(r, \varphi, \theta) &= r \sin\varphi \sin\theta \\ z(r, \varphi, \theta) &= r \cos\theta \end{aligned} \tag{1}$$

where $r \in [0, \infty]$ the radial distance, $\varphi \in [0, 2\pi)$ the azimuthian angle, and $\theta \in [0, \pi]$ the polar angle.

1. (2/10) Show that $\hat{\mathbf{p}}^2 = \frac{1}{r^2}\left[\hat{\mathbf{L}}^2 + (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2 - i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}}\right]$
2. (2/10) Find the spherical unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$ in terms of the cartesian unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$.
3. (2/10) Evaluate the cartesian derivatives of the radial $\partial_\mu r$, the azimuth $\partial_\mu \varphi$, and the polar $\partial_\mu \theta$, (where $\mu = x, y, z$) and find the result in terms of (r, φ, θ).
4. (2/10) Use the previous results to write $\boldsymbol{\nabla} = \mathbf{e}_x \partial_x + \mathbf{e}_y \partial_y + \mathbf{e}_z \partial_z$ in spherical coordinates.
5. (2/10) Find the differential operator form of $\hat{L}_z$, $\hat{L}_+$, $\hat{L}_-$, and $\hat{\mathbf{L}}^2$ in spherical coordinates.

Exercise 2 (10 points)

In the lectures we learned that the spherical harmonics can be written as $Y_{lm}(\varphi, \theta) = A_{ml} P_l^m(\cos\theta) \mathrm{e}^{im\varphi}$, where A_{ml} the normalization constant, and $P_l^m(z)$ the associated Legendre polynomials. We recall the lecture notes, Ch.4, page

11, definition

$$P_l^m(z) = (-1)^m (1-z^2)^{\frac{m}{2}} \partial_z^m P_l(z) = (-1)^m \frac{(1-z^2)^{\frac{m}{2}}}{2^l l!} \partial_z^{l+m} (z^2-1)^l$$

where $P_l(z)$ simpler Legendre polynomials. The Legendre polynomials are solutions to the differential equation

$$(1-z^2)\partial_z^2 P_l(z) - 2z\partial_z P_l(z) + l(l+1)P_l(z) = 0$$

1. (1/10) Find the transformation rule for $Y_{lm}(\varphi,\theta)$ under spacial inversion $\mathbf{r} \to -\mathbf{r}$.
2. (3/10) Show that the associated Legendre polynomials $P_l^m(z)$ are solutions to the second order partial differential equation

$$(1-z^2)\partial_z^2 P_l^m(z) - 2z\partial_z P_l^m(z) + \left[l(l+1) - \frac{m^2}{1-z^2}\right] P_l^m(z) = 0.$$

3. (2/10) Using the differential operator form in spherical coordinates show that $\hat{L}_z Y_{lm}(\varphi,\theta) = m\hbar Y_{lm}(\varphi,\theta)$ and that $\hat{\mathbf{L}}^2 Y_{lm}(\varphi,\theta) = l(l+1)\hbar^2 Y_{lm}(\varphi,\theta)$.

You are given a prepared system described by the wavefunction $\psi(\mathbf{r}) = N\mathrm{e}^{-r/2a} f(\varphi,\theta)$, where N the normalization constant, a a positive constant with units of length, and the angular dependence is given by

$$f(\varphi,\theta) = \frac{1}{\sqrt{\pi}}\cos^2\left(\frac{\theta}{2}\right) - \frac{\sin(2\theta)\cos(\varphi)}{2\sqrt{2\pi}}.$$

In the following you can use that $\int\limits_0^{+\infty} \mathrm{d}r r^2 \mathrm{e}^{-r} = 2$.

4. (2/10) Normalize the wavefunction and find N.
5. (2/10) If we measured once the angular momentum state of the system, which angular momentum states can be measured with non zero probability? Find the expectation values $\langle \hat{L}_x \rangle$, $\langle \hat{L}_y \rangle$, $\langle \hat{L}_z \rangle$.