Exercise Sheet 5 **Theoretische Physik 4: Quantenmechanik**

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Exercise 1 (12 points)

For two operators $\hat{A}$, $\hat{B}$ we want to show the identity

$$\mathrm{e}^{-\hat{A}}\hat{B}\mathrm{e}^{\hat{A}} = \hat{B} + \frac{1}{1!}[\hat{B},\hat{A}] + \frac{1}{2!}[[\hat{B},\hat{A}],A] + ... = \sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{C}_{\hat{A}}^{(n)}(\hat{B}), \tag{1}$$

where, for compact notation, we have defined the operator function $\mathcal{C}_{\hat{A}}(\hat{B}) = [\hat{B},\hat{A}]$. The notation $\mathcal{C}_{\hat{A}}^{(n)}(\hat{B})$ indicates the n^{th} composition of the function. The null composition $n=0$, in other words no function operation, is just the function argument $\mathcal{C}_{\hat{A}}^{(0)}(\hat{B}) = B$. The $n=1$ composition is the function applied once $\mathcal{C}_{\hat{A}}^{(1)}(\hat{B}) = \mathcal{C}_{\hat{A}}(\hat{B}) = [\hat{B},\hat{A}]$. The $n=2$ composition is $\mathcal{C}_{\hat{A}}^{(2)}(\hat{B}) = \mathcal{C}_{\hat{A}}(\mathcal{C}_{\hat{A}}(\hat{B})) = [\mathcal{C}_{\hat{A}}(\hat{B}),\hat{A}] = [[\hat{B},\hat{A}],\hat{A}]$. The pattern continues like this for any n.

1. (2/12) Define the function $f(u) = \mathrm{e}^{-u\hat{A}}\hat{B}\mathrm{e}^{u\hat{A}}$ where $u \in \mathbb{R}$ a real variable. Show that $\partial_u^n f(u) = \mathrm{e}^{-u\hat{A}}\mathcal{C}_{\hat{A}}^{(n)}(\hat{B})\mathrm{e}^{u\hat{A}}$ by induction.
2. (2/12) Use the above result to prove the identity Eq. (1).

Some times the commutators terminate quickly. Lest consider the case where $[\hat{A},\hat{B}] = c$ is just a complex constant $c \in \mathbb{C}$ (we have already met such cases, for example $\hat{x}$ and $\hat{p}$, or $\hat{a}^\dagger$ and $\hat{a}$). Then we have that another important identity to show

$$\text{If } [\hat{A},\hat{B}] = c \in \mathbb{C}, \text{ then } \mathrm{e}^{\hat{A}}\mathrm{e}^{\hat{B}} = \mathrm{e}^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}. \tag{2}$$

3. (2/12) Define the operator function $f(u) = \mathrm{e}^{u\hat{A}}\mathrm{e}^{u\hat{B}}$, where $u \in \mathbb{R}$ a real variable, and assuming $[\hat{A},\hat{B}] = c \in \mathbb{C}$, show that $\partial_u f(u) = (\hat{A} + \hat{B} + uc)f(u)$.
4. (2/12) Find the solution $f(u)$ to the differential equation above, and use it to prove the identity Eq. (2).

Lets consider any system, that has somehow simplified to the hamiltonian $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$, where $\hat{a}^\dagger$, $\hat{a}$ a pair of ladder operators (this can be as simple as the harmonic oscillator, or an emergent harmonic oscillator hamiltonian like we saw last week with the magnetic field). Lets call ψ_n the eigenstates of $\hat{H}$ with energy eigenvalue E_n. One can define a displacement operator in terms of the ladder operators:

$$\hat{D}(z) = \mathrm{e}^{z\hat{a}^\dagger - z^*\hat{a}}, \tag{3}$$

where $z \in \mathbb{C}$ a complex variable. The displacement operator $\hat{D}(z)$ can be shown to be a generator of coherent states.

5. (2/12) Show that $\phi_z = \hat{D}(z)\psi_0$ is a coherent state.
6. (2/12) Show that the operator can be re-written in the form $\hat{D}(z) = \mathrm{e}^{-|z|^2/2}\mathrm{e}^{z\hat{a}^\dagger}\mathrm{e}^{z^*\hat{a}}$, and use it to find the expansion of ϕ_z into the ψ_n states.

Exercise 2 (8 points)

Consider a particle of mass m in the infinite square well potential

$$V(x) = \begin{cases} 0 & |x| \le \frac{L}{2} \\ \infty & |x| > \frac{L}{2} \end{cases}. \tag{4}$$

Lets say we have carefully prepared the system at time $t = 0$ and it is described by the wavefunction

$$\psi(x,0) = \begin{cases} N(x - \frac{L}{2})(x + \frac{L}{2}) & \text{for } |x| \leq \frac{L}{2} \\ 0 & |x| > \frac{L}{2} \end{cases} \tag{5}$$

1. (1/8) Normalize $\psi(x,0)$ and find N .
2. (3/8) Suppose you measure the energy of the particle at $t = 0$. What are the energy values that could be found? What is the probability to measure each energy value? You can make use of the following two integrals:

$$\int \mathrm{d}x\, x^2 \cos x = (x^2 - 2)\sin x + 2x\cos x, \quad \int \mathrm{d}x\, x^2 \sin x = -(x^2 - 2)\cos x + 2x \sin x$$

3. (3/8) Evaluate explicitly the probability to measure the ground state to 4 decimal points, and discuss if this result should be expected. Also discuss the overall result you found above, and if it should have been expected.
4. (1/8) Write down the wave function $\psi(x,t)$ for all times t.