Exercise Sheet 4 **Theoretische Physik 4: Quantenmechanik**

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Exercise 1 (20 points)

Consider a particle of charge q and mass m, in the presence of an external magnetic field $\mathbf{B}$. The Hamiltonian that describes such a system is given by

$$H = \frac{(\mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{r}))^2}{2m} \tag{1}$$

where $\mathbf{p} = m\dot{\mathbf{r}} + \frac{q}{c}\mathbf{A}(\mathbf{r})$ is the generalized momentum corresponding to the generalized position $\mathbf{r}$, and $\mathbf{A}(\mathbf{r})$ is the vector potential that describes the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})$. As we saw in the lectures, the canonical quantization proceeds by promoting $\mathbf{r}$ and $\mathbf{p}$ to operators $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$, such that $[r_\mu, p_\nu] = i\hbar\delta_{\mu\nu}$. Solving the quantum problem of a charged particle in a magnetic field is closely related to the methods of the harmonic oscillator.

Define the mechanical momentum $\boldsymbol{\pi} = m\dot{\mathbf{r}} = \mathbf{p} - \frac{q}{c}\mathbf{A}(\mathbf{r})$. Of course without a magnetic field ($\mathbf{A}(\mathbf{r}) = 0$), the mechanical momentum $\hat{\boldsymbol{\pi}}$ and the generalized momentum $\mathbf{p}$ coincide, so this distinction never arose up to now. Lets figure out the properties of the mechanical momentum.

1. (1/20) As a useful prelude, show that $[\hat{p}_\mu, f(\hat{\mathbf{r}})] = -i\hbar\partial_\mu f(\mathbf{r})$ for any function $f(\hat{\mathbf{r}})$ of the real space position $\mathbf{r}$.

2. (2/20) Show that the commutation relations of the components of the mechanical momentum are $[\hat{\pi}_\mu, \hat{\pi}_\nu] = i\frac{q}{c}\hbar B_\lambda \varepsilon_{\lambda\mu\nu}$.

Lets try and solve a simple problem of a uniform magnetic field. Consider a particle of charge $q > 0$ that is trapped and can only move in the xy-plane, under the influence of an external uniform magnetic field $\mathbf{B} = -B\mathbf{e}_z$ pointing along the $(-z)$-axis. When ever $\mathbf{A}(\mathbf{r})$ is explicitly needed work in the symmetric gauge $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r} \times \mathbf{B}) = \frac{B}{2}(y\mathbf{e}_x - x\mathbf{e}_y)$.

3. (1/20) Show that the symmetric gauge does indeed give the uniform magnetic field along the z-axis.

4. (1/20) Write the Hamiltonian of this system in terms of $\hat{\pi}_x$ and $\hat{\pi}_y$, and discuss whether $\hat{\pi}_x$ and $\hat{\pi}_y$ are good operators to describe the system (think about when do you share eigenvectors between operators?).

5. (2/20) Show that building new operators $\hat{a} = g(\hat{\pi}_x - i\hat{\pi}_y)$, $\hat{a}^\dagger = g(\hat{\pi}_x + i\hat{\pi}_y)$, are ladder operators, where $\hat{a}$ the lowering operator and $\hat{a}^\dagger$ the raising operator, with an appropriate choice of the constant g, and find what is this g.

6. (2/20) Use the ladder operators $\hat{a}$, $\hat{a}^\dagger$ to show that the Hamiltonain is actually very familiar, and find that the energy eigenvalues are $E_n = \hbar\omega_B\left(n + \frac{1}{2}\right)$ for $n \geq 0$, where $\omega_B = \frac{qB}{mc}$.

Next we will find the ground state wavefunction $\psi_0(x, y)$ that corresponds to the lowest energy eigenvalue $E_0 = \hbar\omega_B/2$. This task will be easy if you define complex coordinates, $z = x - iy$ and $\bar{z} = z^* = x + iy$, and their holomorphic derivatives $\partial = \frac{1}{2}(\partial_x + i\partial_y)$ and $\bar{\partial} = \frac{1}{2}(\partial_x - i\partial_y)$. You can easily verify that the holomorphic derivatives are such that $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$. this means that a function of two variables $\psi_0(x, y)$ can be equally thought of a function of a complex variable and its conjugate $\psi_0(z, \bar{z})$, where z and $\bar{z}$ behave as if separate variables under the holomorphic derivatives.

7. (3/20) Given that the lowering operator $\hat{a}$ is defined such that it destroys the ground state $\hat{a}\psi_0 = 0$ (since there is no where lower to go), and working in the symmetric gauge, show that the form for the ground state eigenstate is $\psi_0(z, \bar{z}) \propto f(z)\mathrm{e}^{-|z|^2/4l_B^2}$ for any smooth (holomorphic) function $f(z)$, where $l_B = \sqrt{\frac{\hbar c}{qB}}$ (do not bother with normalizations).

So the two-dimensional system has reduced to a very familiar one-dimensional system, and you are using its language to describe a completely different scenario. But wait, how did we get only one quantum number n out of a two-dimensional system? Note that each spacial dimension should add a quantum number (for example, a running wave in two dimensions $\mathrm{e}^{-ik_x x}\mathrm{e}^{-ik_y y}$ with k_x and k_y two independent quantum numbers, this behaviour persists in localized bound states too). Did we miss something?

8. (3/20) Define additional "momentum like" operators $\hat{\boldsymbol{\tau}} = \hat{\mathbf{p}} + \frac{q}{c}\mathbf{A}$, and show that constructing $\hat{b} = g(\hat{\tau}_x + i\hat{\tau}_y)$, $\hat{b}^\dagger = g(\hat{\tau}_x - i\hat{\tau}_y)$ gives you a new set of ladder operators, where g is the same as determined before for $\hat{a}$, $\hat{a}^\dagger$.
9. (2/20) Show that $[\hat{H}, \hat{\tau}_\mu] = 0$, and from this qualitatively argue about the potential degeneracy of E_n states and what operator could label them.
10. (3/20) Use the new lowering operator $\hat{b}$ to determine $f(z)$ in $\psi_0(z, \bar{z})$ (do not bother with normalizations) and discuss how can you construct all the eigenstates that correspond to E_n states for any n?