Exercise Sheet 0 **Theoretische Physik 4: Quantenmechanik**

SoSe 2025 Prof. Dr. C. Gros

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Comments on procedure: There will be one exercise sheet per week, each sheet will be worth 20 points total. The exercise sheet will be posted on Monday, and solutions must be returned by the next Monday noon. If there are special circumstances that alter this schedule, you will see it at the top of the exercise sheet under "Posted ... due by ...".

The solutions prepared by the students are to be handed in electronically by uploading to OLAT, either scans of hand written solutions or electronically prepared solutions. In either case the text must be clear and legible. Corrected assignments will be returned by the respective tutors.

When making solutions, always write your name and student numbers at the very top of the first page. Each solution needs to be clearly numbered ("Ex 1.1", "Ex 1.2", ..., "Ex 2.1", ... etc.). If using already known equations or facts not explicitly mentioned in the exercise, clearly refer to them (for example "as in lecture notes , equation # on page #" or "as per the statement in lecture notes page #").

Exercise 1 (5 points)

Consider a general 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{1}$$

with complex number entries $a_{ij} \in \mathbb{C}$.

1. (2/5) Find the eigenvalues and eigenvectors of the matrix Eq.(1) (do not bother with the normalization of the eigenvectors).

Observables are described by hermitian operators, and their corresponding matrices, which are hermitian matrices, describe their action with respect to some chosen basis. Consider the case where A is a 2×2 hermitian matrix $A = A^\dagger = (A^*)^T$.

2. (1/5) What constrains does this place on the matrix elements of Eq.(1)? Is there a relation between the a_{ij}?
3. (1/5) Are the eigenvalues of the hermitian matrix real, imaginary, or general complex?
4. (1/5) Are the eigenvectors orthogonal?

Exercise 2 (5 points)

While observables are described by hermitian matrices, spatial transformations (like rotation and translation) are described by unitary matrices. Consider the case where A is a 2×2 unitary matrix $AA^\dagger = A^\dagger A = \mathbb{1}$. This should again place constrains on the matrix form just like in the previous exercise.

1. (2/5) Starting from Eq.(1), show that we can write the matrix in the form

$$\begin{pmatrix} a_{11} & a_{12} \\ -a_{12}^* e^{i\vartheta} & a_{11}^* e^{i\vartheta} \end{pmatrix}, \ |a_{11}|^2 + |a_{12}|^2 = 1, \vartheta \in \mathbb{R}. \tag{2}$$

   (Hint: Using the properties of the determinant, show from $AA^\dagger = \mathbb{1}$ that $\det(A) = e^{i\vartheta}$ with $\vartheta \in \mathbb{R}$. Then use the element wise equality $A^\dagger = A^{-1}$)

2. (2/5) Are the eigenvalues of the unitary matrix real, imaginary, or general complex?
3. (1/5) What special relation do the two eigenvalues have when $\vartheta = 0$?

Exercise 3 (5 points)

We are given a unitary or hermitian matrix A of arbitrary size $n \times n$. We can in principle find the eigenvalues and eigenvector that satisfy the eigenvalue equation $A\vec{u}_i = \alpha_i \vec{u}_i$.

1. (1/5) What is the number of eigenvectors $\vec{u}_i$ we must have?
2. (1/5) What is the maximum number of eigenvalues a_i we can possibly have?
3. (1/5) What is the significance of having less eigenvalues than the maximum possible?

Matrices generally need not commute, so generally given a random matrix B we will have $[A, B] = AB - BA \neq 0$. Consider the special case of a diagonalizable matrix B that happens to commutes with A, or in other words $[A, B] = 0$. Let us assume that the spectrum (eigenvalues) of A and B matrices is non-degenerate, or in other words all eigenvalues are a different value and each corresponds to only one eigenvector.

4. (2/5) Starting from the eigenvalue equation $A\vec{u}_i = \alpha_i \vec{u}_i$ show that matrix A and B are sharing eigenvectors, or in other words that they have the same eigenvectors (this property will hold in the degenerate case as well).

Exercise 4 (5 points)

Consider the set of real-space wavefunctions

$$f_n(x) = \begin{cases} A_n \cos\left(\frac{n\pi x}{L}\right), & x \in [-L, L] \\ 0 \end{cases} \tag{3}$$

where $n \in \mathbb{Z}$ integer, $L > 0$, and A_n a normalization constants.

1. (1/5) Using the normalization condition $\|f_n(x)\| = \int_{-\infty}^{\infty} \mathrm{d}x |f_n(x)|^2 = 1$, show that the constants A_n must be equal to $A_n = 1/\sqrt{L}$.
2. (1/5) Lets figure out if this set of functions can form a basis, by checking their orthogonality under the inner product $\langle f_n(x), f_m(x) \rangle = \int_{-\infty}^{\infty} \mathrm{d}x f_n(x)^* f_m(x)$: Given two different values $n, m \in \mathbb{Z}$, $n \neq m$, check that $f_n(x)$ and $f_m(x)$ are orthogonal by evaluating the inner product $\langle f_n(x), f_m(x) \rangle$.
3. (2/5) For every real-space wavefunction $f_n(x)$ we can find the corresponding momentum-space wavefunction $\tilde{f}_n(k)$, by using the Fourier transform $\tilde{f}_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ikx} f(x)$. Show that for our case :

$$\tilde{f}_n(k) = (-1)^n \sqrt{\frac{2L}{\pi}} \frac{kL \sin(kL)}{(kL)^2 - (n\pi)^2}$$

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4. (1/5) If we wanted a single $f_n(x)$ to describe some form of standing wave in $x \in [-L, L]$ would that be possible? (Hint: What condition does $f_n(L)$ and $f_n(-L)$ must fulfil to have a standing wave, and is it fulfilled here?)