Chapter 16

Self-Organized Criticality

Self-organized criticality. Normal physical and dynamical systems show criticality only for selected parameters, e.g. right at the transition temperature $T = T_c$. For criticality to be biologically relevant, the system must evolve into a critical state starting from a wide range of initial states – one speaks of 'self-organized criticality'. This phenomenon is an example of how the concepts of statistical mechanics can be applied outside the classical phyical setting.

16.1 The sandpile model

The sandpile model. Per Bak and coworkers introduced a simple cellular automaton which mimics the properties of sandpiles, the BTW-model. Every cell is characterized by a force

$$z_i \;=\; z(x,y) \;=\; 0,\ 1,\ 2,\ \ldots, \qquad\qquad x,y = 1,\ldots,L$$

on a finite $L \times L$ lattice. There is no one-to-one correspondence of the sandpile model to real-world sandpiles. Loosely speaking one may identify the force z_i with the slope of real-world sandpiles. But this analogy is not rigorous, as the slope of a real-world sandpile is a continuous variable. The slopes belonging to two neighboring cells should therefore be similar, whereas the values of z_i and z_j on two neighboring cells can differ by an arbitrary amount within the sandpile model.
The sand begins to topple when the slope gets too big:

$$z_j \;\to\; z_j - \Delta_{ij}, \qquad\qquad \text{if} \qquad z_j > K\ ,$$

where K is the threshold slope and with the toppling matrix

$$\Delta_{i,j} \;=\; \left\{ \begin{array}{rll} 4 & i=j & \\ -1 & i,j & \text{nearest neighbors} \\ 0 & & \text{otherwise} \end{array} \right. . \tag{16.1}$$

This update-rule is valid for the 4-cell neighborhood $\{(0,\pm1),(\pm1,0)\}$. The threshold K is arbitrary, a shift in K simply shifts the z_i. It is custom to consider $K = 3$. Any initial random configuration will then relax into a steady-state final configuration (called stable state) with

$$z_i \;=\; 0,\; 1,\; 2,\; 3, \qquad\qquad \text{(stable-state)}\ .$$

Open boundary conditions. The update-rule (16.1) is conserving:

> *If there is a quantity which is not changed by the update rule it is said to be conserving.*

The sandpile model is locally conserving. The total height $\sum_j z_j$ is constant due to $\sum_j \Delta_{i,j} = 0$. Globally though it is not conserving, as one uses open boundary conditions for which excess sand is lost at the boundary. When a site at the boundary topples, some sand is lost there and the total $\sum_j z_j$ is reduced by one.
However, here is only a vague relation of the BTW-model to real-world sandpiles. The conserving nature of the sandpile model mimics the fact that sand grains cannot be lost in real-world sandpiles. This interpretation contrasts however with the previously assumed correspondence of the z_i with the slope of real-world sandpiles.

Avalanches. When starting from a random initial state with $z_i \ll K$ the system settles in a stable configuration when adding 'grains of sand' for a while. When a sandcorn is added to a site with $z_i = K$

$$z_i \;\rightarrow\; z_i + 1, \qquad\qquad z_i = K\ ,$$

a toppling event is induced, which may in turn lead to a whole series of topplings. The resulting avalanche is characterized by its duration t and the size s of affected sites. It continues until a new stable configuration is reached.

Step 1

3	0	3	2	3
1	2	3	1	0
2	3	3+1	1	3
3	1	0	2	0
0	2	1	3	1

Step 2

3	0	3	2	3
1	2	4	1	0
2	4	0	2	3
3	1	1	2	0
0	2	1	3	1

Step 3

3	0	4	2	3
1	4	0	2	0
3	0	2	2	3
3	2	1	2	0
0	2	1	3	1

Step 4

3	2	0	3	3
2	0	2	2	0
3	1	2	2	3
3	2	1	2	0
0	2	1	3	1

Distribution of avalanches. We define with $D(s)$ and $D(t)$ the distributions of the size and of the duration of avalanches. One finds that they are scale free,

$$D(s) \;\sim\; s^{-\alpha_s}, \qquad\qquad D(t) \;\sim\; t^{-\alpha_t}\ , \tag{16.2}$$

which is the tell-tale sign of self-organized criticality. We expect these scale-free relations to be valid for a wide range of cellular automata with conserving dynamics, independent of the special values of the parameters entering the respective update functions. Numerical simulations and analytic approximations yield for $d = 2$ dimensions

$$\alpha_s \;\approx\; \frac{5}{4}, \qquad\qquad \alpha_t \;\approx\; \frac{3}{4}\ .$$

Conserving dynamics and self-organized criticality. We note that the toppling events of an avalanche are (locally) conserving. Avalanches of arbitrary large sizes must therefore occur, as sand can be lost only at the boundary of the system. One can indeed prove that Eqs. (16.2) are valid only for locally conserving models. Self-organized criticality breaks down as soon as there is a small but non-vanishing probability to loose sand somewhere inside the system.

Features of the critical state. The empty board, when all cells are initially empty, $z_i \equiv 0$, is not critical. The system remains in the frozen phase when adding sand, as long as most $z_i < K$. Adding one sand corn after the other the critical state is slowly approached. There is no way to avoid the critical state.
Once the critical state is achieved the system remains critical. This critical state is paradoxically also the point at which the system is dynamically most unstable. It has an unlimited susceptibility to an external driving (adding a sandcorn), as a single added sandcorn can trip avalanches of arbitrary size.
It needs to be noted that the dynamics of the sandpile model is deterministic, once the sandcorn has been added, and that the disparate fluctuations in terms of induced avalanches are features of the critical state per se and not due to any hidden stochasticity, or due to any hidden deterministic chaos.

16.2 Probability Generating Function Formalism

For the treatement of the sandpile model we will need to handle the probabilites determining the length and the duration of an avalanche. For this purpose we will use a powerful method from probability theory, namely the generating function formalism.

Probability Generating Functions. We define by

$$G_0(x) = \sum_{k=0}^{\infty} p_k x^k \tag{16.3}$$

the *generating function* $G_0(x)$ for the probability distribution p_k. The generating function $G_0(x)$ contains all information present in p_k. We can recover p_k from $G_0(x)$ simply by differentiation:

$$p_k = \frac{1}{k!} \frac{\mathrm{d}^k G_0}{\mathrm{d}x^k}\bigg|_{x=0} . \tag{16.4}$$

One says that the function G_0 "generates" the probability distribution p_k.

Properties of Generating Functions. Probability generating functions have a couple of important properties:

1. Normalization: The distribution p_k is normalized and hence

$$G_0(1) = \sum_k p_k = 1 . \tag{16.5}$$

2. Mean: A simple differentiation

$$G_0'(1) = \sum_k k\, p_k = \langle k \rangle \tag{16.6}$$

yields the average degree $\langle k \rangle$.

3. Moments: The nth moment $\langle k^n \rangle$ of the distribution p_k is given by

$$\langle k^n \rangle = \sum_k k^n p_k \;=\; \left[\left(x \frac{\mathrm{d}}{\mathrm{d}x} \right)^n G_0(x) \right]_{x=1} . \tag{16.7}$$

The Generating Function for Independent Random Variables. Let us assume that we have two random variables. As an example we consider two dice. Throwing the two dice are two independent random events. The joint probability to obtain $k = 1, \dots, 6$ with the first die and $l = 1, \dots, 6$ with the second dice is $p_k\, p_l$. This probability function is generated by

$$\sum_{k,l} p_k p_l\, x^{k+l} = \left(\sum_k p_k\, x^k \right) \left(\sum_l p_l\, x^l \right) ,$$

i.e. by the product of the individual generating functions. This is the reason why generating functions are so useful in describing combinations of independent random events. As an application consider n randomly chosen vertices. The sum $\sum_i k_i$ of the respective degrees has a cumulative degree distribution, which is generated by

$$\Big[G_0(x) \Big]^n .$$

The Generating Function of the Poisson Distribution. As an example we consider the Poisson distribution $p_k = e^{-z}\, z^k / k!$, with z being the average degree. Using Eq. (16.3) we obtain

$$G_0(x) = e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{k!}\, x^k = e^{z(x-1)} . \tag{16.8}$$

This is the generating function for the Poisson distribution.

Stochastic Sum of Independent Variables. Let's assume we have random variables $k_1,\ k_2, \dots,$ each having the same generating functional $G_0(x)$. Then

$$G_0^2(x), \qquad G_0^3(x), \qquad G_0^4(x), \ \dots$$

are the generating functionals for

$$k_1 + k_2, \qquad k_1 + k_2 + k_3, \qquad k_1 + k_2 + k_3 + k_4, \ \dots \ .$$

Now consider that the number of times n this stochastic process is executed is distributed as p_n. As an example consider throwing a dice several times, with a probability p_n of throwing exactly n times. The distribution of the results obtained is then generated by

$$\sum_n p_n G_0^n(x) = G_N\left(G_0(x)\right), \qquad G_N(z) = \sum_n p_n z^n . \tag{16.9}$$

16.3 Random branching theory

Branching theory deals with the growth of networks via branching. Networks generated by branching processes are loopless, they typically arise in theories of evolutionary processes. Avalanches have an intrinsic relation to branching processes: At every time step the avalanche can either continue or stop.

Branching in sandpiles. A typical update during an avalanche is of the form

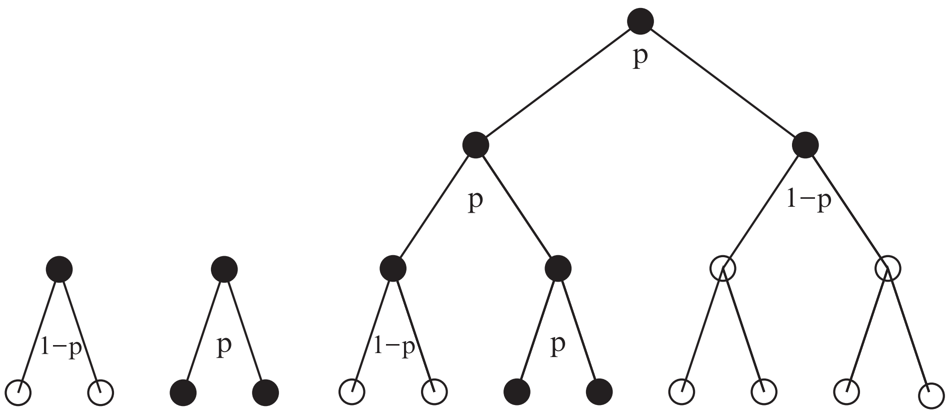
$$\begin{array}{lll} \text{time 0:} & z_i \to z_i - 4 & z_j \to z_j + 1 \\ \text{time 1:} & z_i \to z_i + 1 & z_j \to z_j - 4 \ , \end{array}$$

when two neighboring cells i and j have initially $z_i = K+1$ and $z_j = K$. This implies that an avalanche typically intersects with itself. Consider however a general d-dimensional lattice with $K = 2d - 1$. The self-interaction of the avalanche (that is the formation of loops) becomes unimportant in the limit $1/d \to 0$ and the avalanche can be mapped rigorously to a random branching process.

Binary random branching. In $d \to \infty$ the notion of neighbors looses meaning, avalanches then have no spatial structure. Every toppling event affects $2d$ neighbors, on a d-dimensional hypercubic lattice. However, only the cumulative probability of toppling of the affected cells is relevant, due to the absence of geometric constraints in the limit $d \to \infty$. All what is important then is the question whether an avalanche continues, increasing its size continously or whether it stops.
We can therefore consider the case of binary branching, *viz* that a toppling event creates two new active sites.

> *One speaks of 'binary branching' if an an active site of an avalanche topples with the probability p and creates 2 new active sites.*

For $p < 1/2$ the number of new active sites decreases on the average and the avalanche dies out. $p_c = 1/2$ is the critical state with (on the average) conserving dynamics.



Distribution of avalanche sizes. The properties of avalanches are determined by the probability distribution,

$$P_n(s,p), \qquad \sum_{s=1}^{\infty} P_n(s,p) = 1 \ ,$$

describing the probability to find an avalanche of size s in a branching process of order n. Here s is the (odd) number of sites inside the avalanche.

Generating function formalism. We introduced in Sect. 16.2 the generating functions for probability distribution. This formalism is very useful when one has to deal with independent stochastic processes, as the joint probability of two independent stochastic processes is equivalent to the simple multiplication of the corresponding generating functions.
We define via

$$f_n(x,p) \;=\; \sum_s P_n(s,p)\,x^s, \qquad f_n(1,p) = \sum_s P_n(s,p) = 1 \tag{16.10}$$

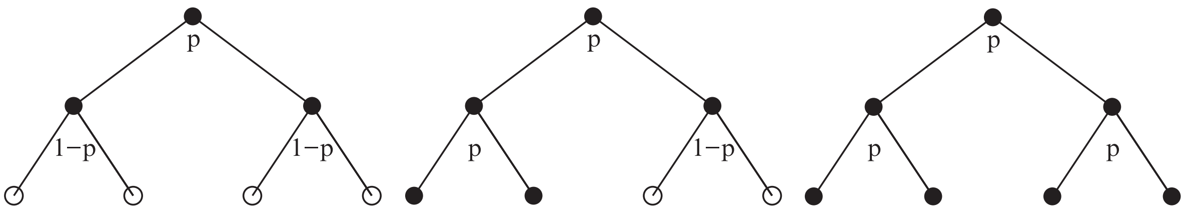
the generating functional $f_n(x,p)$ for the probability distribution $P_n(s,p)$. We note that

$$P_n(s,p) \;=\; \frac{1}{s!}\,\frac{\partial^s f_n(x,p)}{\partial x^s}\Big|_{x=0}, \qquad n,\ p\ \text{ fixed}\ . \tag{16.11}$$

Small avalanches. For small s and large n one can evaluate the probability for small avalanches to occur by hand and one finds for the corresponding generating functionals:

$$P_n(1,p) = 1-p, \qquad P_n(3,p) = p(1-p)^2, \qquad P_n(5,p) = 2p^2(1-p)^3\ .$$

Note that $P_n(1,p)$ is the probability to find an avalanche of just one site.



Recursion relation. For generic n the recursion relation

$$f_{n+1}(x,p) \;=\; x\,(1-p) \,+\, x\,p\,f_n^2(x,p) \tag{16.12}$$

is valid. To see why, one considers building the branching network backwards, adding a site at top:

- With the probability $(1-p)$
  one adds a single-site avalanche described by the generating functional x.

- With the probability p
  one adds a site, described by the generating functional x, which generated two active sites, described each by the generating functional $f_n(x,p)$.

Self-consistency condition. For large n and finite x the generating functionals $f_n(x,p)$ and $f_{n+1}(x,p)$ become identical, leading to the self-consistency condition

$$f_n(x,p) \;=\; f_{n+1}(x,p) \;=\; x\,(1-p) \,+\, x\,p\,f_n^2(x,p)\ , \tag{16.13}$$

with the solution

$$f(x,p) \;\equiv\; f_n(x,p) \;=\; \frac{1-\sqrt{1-4x^2p(1-p)}}{2xp} \tag{16.14}$$

for the generating functional $f(x,p)$. The normalization condition

$$f(1,p) \;=\; \frac{1-\sqrt{1-4^2p(1-p)}}{2p} \;=\; \frac{1-\sqrt{(1-2p)^2}}{2p} \;=\; 1$$

is fulfilled for $p \in [0,1/2]$. For $p > 1/2$ the last step in above equation would not be correct.

Subcritical solution. Expanding Eq. (16.14) in powers of x^2 we find terms like

$$\frac{1}{p}\Big[4p(1-p)\Big]^k \frac{(x^2)^k}{x} \;=\; \frac{1}{p}\Big[4p(1-p)\Big]^k x^{2k-1}\ .$$

Comparing with the definition of the generating functional (16.10) we note that $s = 2k-1$, $k = (s+1)/2$ and that

$$P(s,p) \;\sim\; \frac{1}{p}\sqrt{4p(1-p)}\,\Big[4p(1-p)\Big]^{s/2} \;\sim\; e^{-s/s_c(p)}\ , \tag{16.15}$$

where we have used the relation

$$a^{s/2} \;=\; e^{\ln(a^{s/2})} \;=\; e^{-s(\ln a)/(-2)}, \qquad\qquad a = 4p(1-p)\ ,$$

and where we have defined the avalanche correlation size

$$s_c(p) \;=\; \frac{-2}{\ln[4p(1-p)]}, \qquad\qquad \lim_{p\to 1/2} s_c(p) \;\to\; \infty\ .$$

For $p < 1/2$ the size-correlation length $s_c(p)$ is finite and the avalanche is consequently not scale free. The characteristic size of an avalanche $s_c(p)$ diverges for $p \to p_c = 1/2$. Note that $s_c(p) > 0$ for $p \in]0,1[$.

Critical solution. We now consider the critical case with

$$p = 1/2, \qquad\qquad 4p(1-p) \;=\; 1, \qquad\qquad f(x,p) \;=\; \frac{1-\sqrt{1-x^2}}{x}\ .$$

The expansion of $\sqrt{1-x^2}$ with respect to x is

$$\sqrt{1-x^2} \;=\; \sum_{k=0}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!}\Big(-x^2\Big)^k$$

in Eq. (16.14) and therefore

$$P_c(k) \;\equiv\; P(s=2k-1, p=1/2) \;\sim\; \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!}(-1)^k\ .$$

This expression is still unhandy. We are however only interested in the asymptotic behavior for large avalanche sizes s. For this purpose we consider the recursive relation

$$P_c(k+1) \;=\; \frac{1/2-k}{k+1}(-1)P_c(k) \;=\; \frac{1-1/(2k)}{1+1/k}P_c(k)$$

in the limit of large $k=(s+1)/2$, where $1/(1+1/k)\approx 1-1/k$,

$$P_c(k+1) \;\approx\; \Big[1-1/(2k)\Big]\Big[1-1/k\Big]\,P_c(k) \;\approx\; \Big[1-3/(2k)\Big]\,P_c(k)\ .$$

This asymptotic relation leads to

$$\frac{P_c(k+1)-P_c(k)}{1} \;=\; \frac{-3}{2k}\,P_c(k), \qquad\qquad \frac{\partial P_c(k)}{\partial k} \;=\; \frac{-3}{2k}\,P_c(k)\ ,$$

with the solution

$$P_c(k) \;\sim\; k^{-3/2}, \qquad D(s) \;=\; P_c(s) \;\sim\; s^{-3/2}, \qquad \alpha_s \;=\; \frac{3}{2}\ , \tag{16.16}$$

for large k,s, since $s=2k-1$.

Distribution of relaxation times. The distribution of the duration n of avalanches can be evaluated in a similar fashion. For this purpose one considers the probability distribution function

$$Q_n(\sigma,p)$$

for an avalanche of duration n to have σ cells at the boundary.

One can then derive a recursion relation analogous to Eq. (16.12) for the corresponding generating functional and solve it self-consistently. We leave this as an exercise for the reader.

The distribution of avalanche durations is then given by considering $Q_n=Q_n(\sigma=0,p=1/2)$, i.e. the probability that the avalanche stops after n steps. One finds

$$Q_n \;\sim\; n^{-2}, \qquad D(t) \;\sim\; t^{-2}, \qquad \alpha_t \;=\; 2\ . \tag{16.17}$$

Tuned or self-organized criticality? The random branching model discussed in this section had only one free parameter, the probability p. This model is critical only for $p\to p_c=1/2$, giving rise to the impression, that one has to fine-tune the parameters in order to obtain criticality, just like in ordinary phase transitions.

This is however not the case. As an example we could generalize the sandpile model to continuous forces $z_i\in[0,\infty]$ and to the update-rules

$$z_i \;\to\; z_i \;-\; \Delta_{ij}, \qquad \text{if} \qquad z_i>K\ ,$$

and

$$\Delta_{i,j} \;=\; \begin{cases} K & i=j \\ -c\,K/4 & i,j \qquad \text{nearest neighbors} \\ -(1-c)\,K/8 & i,j \qquad \text{next-nearest neighbors} \\ 0 & \qquad\;\; \text{otherwise} \end{cases} \tag{16.18}$$

for a square-lattice with 4 nearest neighbors and 8 next-nearest neighbors (Manhattan distance). The update-rules are conserving,

$$\sum_j \Delta_{ij} \;=\; 0, \qquad\qquad \forall c \in [0,1] \;.$$

For $c = 1$ it corresponds to the continuous-field generalization of the BTW-model. The model defined by Eqs. (16.18), which has not been studied in literature yet, might be expected to map in the limit $d \to \infty$ to an appropriate random branching model with $p = p_c = 1/2$ and to be critical for all values of the parameters K and c, due to its conserving dynamics.