

Chapter 13

Ideal Fermi gas

The properties of an ideal Fermi gas are strongly determined by the Pauli principle. We shall consider the limit:

$$k_B T \ll \mu, \quad \beta\mu \gg 1,$$

which defines the *degenerate Fermi gas*. In this limit, the quantum mechanical nature of the system becomes especially important, and the system has little to do with the classical ideal gas.

Since this chapter is devoted to fermions, we shall omit in the following the subscript $(-)$ that we used for the fermionic statistical quantities in the previous chapter.

13.1 Equation of state

Consider a gas of N non-interacting fermions, e.g., electrons, whose one-particle wavefunctions $\varphi_r(\vec{r})$ are plane-waves. In this case, a complete set of quantum numbers r is given, for instance, by the three cartesian components of the wave vector \vec{k} and the z spin projection m_s of an electron:

$$r \equiv (k_x, k_y, k_z, m_s).$$

Spin-independent Hamiltonians. We will consider only *spin independent* Hamiltonian operator of the type

$$\hat{H} = \sum_{\mathbf{k}} \epsilon_k c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \int d^3r V(\mathbf{r}) c_{\mathbf{r}}^\dagger c_{\mathbf{r}},$$

where the first and the second terms are respectively the kinetic and the potential energy. The summation over the states r (whenever it has to be performed) can then be reduced to the summation over states with different wavevector \mathbf{k} ($\mathbf{p} = \hbar\mathbf{k}$):

$$\sum_r \dots \Rightarrow (2s+1) \sum_{\mathbf{k}} \dots,$$

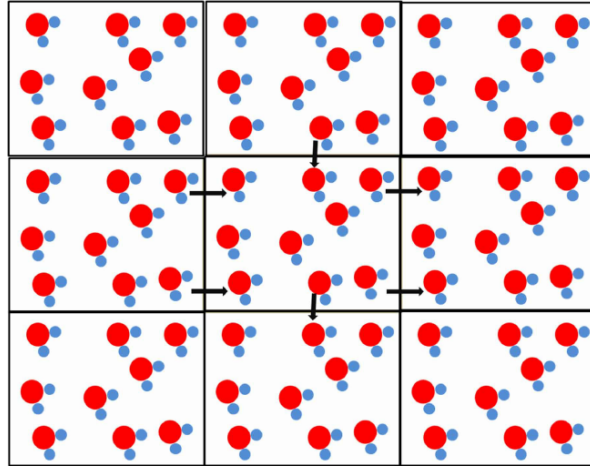
where the summation over the spin quantum number $m_s = -s, -s+1, \dots, s$ has been taken into account by the prefactor $(2s+1)$.

Wavefunctions in a box. We assume that the electrons are in a volume defined by a cube with sides L_x , L_y , L_z and volume $V = L_x L_y L_z$. For the one-particle wavefunction

$$\langle \mathbf{r} | \mathbf{k} \rangle = \psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k} \cdot \mathbf{r}}$$

we use periodicity condition, here along the x -direction, at the cube's walls,

$$e^{ik_x x} = e^{ik_x x + ik_x L_x},$$



which is then translated into a condition for the allowed k -values:

$$e^{ik_x L_x} = e^{i2\pi n_x}, \quad k_x = \frac{2\pi}{L_x} n_x, \quad n_x \in \mathbf{Z}.$$

Analogously for the y - and for the z direction.

Summation over wavevectors. Each state has in k -space an average volume of

$$\Delta k = \frac{(2\pi)^3}{L_x L_y L_z} = \frac{(2\pi)^3}{V}. \quad (13.1)$$

For large $V \rightarrow \infty$ we can then replace the sum \sum_r over all quantum number by

$$\begin{aligned} \sum_r &\rightarrow (2s+1) \frac{1}{\Delta k} \int d^3 k = (2s+1) \frac{V}{(2\pi)^3} \int d^3 k \\ &= (2s+1) \frac{V}{h^3} \int d^3 p, \end{aligned} \quad (13.2)$$

where $k = p/\hbar$ has been used.

The factor $\frac{1}{h}$ ($\frac{1}{h^{3N}}$ for N particles) introduced “ad hoc” in classical statistical physics in Sect. 8.2 appears naturally in quantum statistical physics. It is a direct consequence of the fact that particles correspond to wavefunctions.

13.1.1 Grand canonical potential

We consider now the expression (12.36) for the fermionic grand canonical potential $\Omega(T, V, \mu)$ that we derived in Sect. 12.5,

$$\Omega(T, V, \mu) = -k_B T \sum_r \ln [1 + e^{-\beta(\epsilon_r - \mu)}].$$

Using the substitution (13.2) and $\int d^3k = 4\pi \int k^2 dk$ we write the grand canonical potential as

$$-\beta \Omega(T, V, \mu) = (2s + 1) \frac{V}{(2\pi)^3} 4\pi \int_0^\infty dk k^2 \ln \left[1 + z e^{-\beta \hbar^2 k^2 / (2m)} \right], \quad (13.3)$$

where we used the usual expressions

$$z = e^{\beta\mu}, \quad \epsilon_r \rightarrow \epsilon_k = \frac{\hbar^2 k^2}{2m}$$

for the fugacity z and for the one-particle dispersion and used an explicit expression for the one-particle energies for free electrons ϵ_k .

Dimensionless variables. Expression (13.3) is transformed further by introducing with

$$x = \hbar k \sqrt{\frac{\beta}{2m}}, \quad k^2 dk = \left(\frac{2m}{\beta \hbar^2} \right)^{3/2} x^2 dx$$

a dimensionless variable x . One obtains

$$-\beta \Omega(T, V, \mu) = (2s + 1) \frac{4V}{\sqrt{\pi}} \left(\frac{m}{2\pi\beta\hbar^2} \right)^{3/2} \int_0^\infty x^2 dx \ln \left(1 + z e^{-x^2} \right).$$

De Broglie wavelengths. By making use of the definition of the thermal de Broglie wavelength λ ,

$$\lambda = \sqrt{\frac{2\pi\beta\hbar^2}{m}},$$

we then get

$$-\beta \Omega(T, V, \mu) = \frac{(2s + 1) 4V}{\lambda^3 \sqrt{\pi}} \int_0^\infty dx x^2 \ln \left(1 + z e^{-x^2} \right). \quad (13.4)$$

Term by term integration. We use the Taylor expansion of the logarithm,

$$\ln(1 + y) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{y^n}{n},$$

in order to evaluate the integral

$$\begin{aligned} \int_0^\infty x^2 dx \ln \left(1 + z e^{-x^2} \right) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} \int_0^\infty dx x^2 e^{-nx^2} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} \left(-\frac{d}{dn} \int_0^\infty dx e^{-nx^2} \right) \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} \left(-\frac{d}{dn} \frac{1}{2} \sqrt{\pi} \frac{1}{\sqrt{n}} \right) \end{aligned}$$

term by term. The result is

$$\int_0^\infty x^2 dx \ln(1 + z e^{-x^2}) = \frac{\sqrt{\pi}}{4} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^{5/2}}.$$

Grand canonical potential. Defining

$$f_{5/2}(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^{5/2}} = \frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \ln(1 + z e^{-x^2}) \quad (13.5)$$

we obtain

$$\beta \Omega(T, V, \mu) = -\frac{2s+1}{\lambda^3} V f_{5/2}(z) \quad (13.6)$$

for the grand canonical potential for an ideal Fermi gas.

Pressure. Our result (13.6) reduces with $\Omega = -PV$ to

$$\frac{P}{k_B T} = \frac{2s+1}{\lambda^3} f_{5/2}(z), \quad \lambda = \sqrt{\frac{2\pi\hbar^2}{k_B T m}}, \quad z = e^{\mu/(k_B T)}, \quad (13.7)$$

which yields the pressure $P = P(T, \mu)$.

Density. With

$$\Omega = -k_B T \ln \mathcal{Z} = -PV \quad \frac{PV}{k_B T} = \ln \mathcal{Z}$$

we find, compare Eq. (10.14),

$$\langle \hat{N} \rangle = z \left(\frac{\partial}{\partial z} \ln \mathcal{Z} \right)_{T,V} = V z \left(\frac{\partial}{\partial z} \frac{P}{k_B T} \right)_{T,V} \quad (13.8)$$

for the number of particles N . The density $n(T, \mu) = \langle \hat{N} \rangle / V$ is then given by

$$n = \frac{\langle \hat{N} \rangle}{V} = \frac{2s+1}{\lambda^3(T)} f_{3/2}(z). \quad (13.9)$$

where we have defined

$$f_{3/2}(z) = z \frac{d}{dz} f_{5/2}(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^{3/2}} \quad (13.10)$$

Thermal equation of state. As a matter of principle one could solve (13.9) for the fugacity $z = z(T, n)$, which could then be used to substitute the fugacity in (13.7) for T and n , yielding such the thermal equation of state. This procedure can however not be performed in closed form.

Rewriting the particle density. The function $f_{3/2}(z)$ entering the expression (13.9) for the particle density may be cast into a different form. Helping is here the result (13.7) for $P(T, \mu)$:

$$\frac{P}{k_B T} = \frac{2s+1}{\lambda^3} f_{5/2}(z), \quad f_{5/2}(z) = \frac{\lambda^3}{2s+1} \frac{\ln \mathcal{Z}}{V}, \quad \frac{PV}{k_B T} = \ln \mathcal{Z},$$

which leads to

$$f_{3/2}(z) = z \frac{d}{dz} f_{5/2}(z) = z \frac{d}{dz} \left(\frac{\lambda^3}{2s+1} \frac{\ln \mathcal{Z}}{V} \right)$$

With

$$\frac{d}{dz} = \frac{d}{d\beta} \left(\frac{d\beta}{dz} \right) = \frac{1}{\mu z} \frac{d}{d\beta}, \quad \beta = \frac{\ln z}{\mu} \quad (13.11)$$

and $\lambda = \sqrt{(2\pi\beta\hbar^2)/m}$ we then find

$$\mu V (2s+1) f_{3/2}(z) = \frac{d}{d\beta} (\lambda^3 \ln \mathcal{Z}) = \frac{3\lambda^3}{2\beta} \ln \mathcal{Z} + \lambda^3 \frac{d}{d\beta} \ln \mathcal{Z}.$$

For the particle density (13.9) we finally obtain

$$n = \frac{2s+1}{\lambda^3(T)} f_{3/2}(z), \quad \boxed{\mu n V = \frac{3}{2} PV + \frac{d}{d\beta} \ln \mathcal{Z}}, \quad n V = N, \quad (13.12)$$

where we have used $\ln \mathcal{Z}/\beta = PV$.

Caloric equation of state. The expression for $\mu n V = \mu N$ in (13.12) leads with

$$U = -\frac{d}{d\beta} \ln \mathcal{Z} + \mu \langle \hat{N} \rangle, \quad \mathcal{Z} = \sum_r e^{-\beta(\epsilon_r - \mu)}.$$

to the caloric equation of state

$$\boxed{U = \frac{3}{2} PV}. \quad (13.13)$$

The equation $U = 3PV/2$ is also valid for the classical ideal gas, as discussed in Sect. 8.2, but it is not anymore valid for relativistic fermions.

13.2 Classical limit

Starting from the general formulas (13.7) for $P(T, \mu)$ and (13.9) for $n(T, \mu)$, we first investigate the classical limit (i.e. the non-degenerate Fermi gas), which corresponds, as discussed in Chap. 11, to

$$\boxed{n \lambda^3 \ll 1}, \quad z = e^{\beta\mu} \ll 1.$$

Under this condition, the Fermi-Dirac distribution function reduces to the Maxwell-Boltzmann distribution function:

$$\langle \hat{n}_r \rangle = \frac{1}{z^{-1}e^{\beta\epsilon_r} + 1} \approx ze^{-\beta\epsilon_r} .$$

Expansion in the fugacity. For a small fugacity z we may retain in the series expansion for $f_{5/2}(z)$ and $f_{3/2}(z)$, compare (13.5) and (13.10), the first terms:

$$\begin{aligned} f_{5/2}(z) &\approx z - \frac{z^2}{2^{5/2}} & \beta P \lambda^3 &\approx (2s+1)z \left(1 - \frac{z}{2^{5/2}}\right) \\ f_{3/2}(z) &\approx z - \frac{z^2}{2^{3/2}} & n \lambda^3 &\approx (2s+1)z \left(1 - \frac{z}{2^{3/2}}\right) \end{aligned} \quad (13.14)$$

where we have used (13.7) and (13.9) respectively.

High-temperature limit. The expression for $n\lambda^3$ in (13.14) reduces in lowest approximation to

$$z^{(0)} \approx \frac{n\lambda^3}{2s+1}, \quad n\lambda^3 \ll 1 . \quad (13.15)$$

The number of particles $n\lambda^3$ in the volume spanned by the Broglie wavelength $\lambda \sim 1/\sqrt{T}$ is hence small are small. This is the case at elevated temperatures.

Fugacity expansion. Expanding in the fugacity $z = \exp(\beta\mu)$

$$z \approx \underbrace{\frac{n\lambda^3}{2s+1}}_{z^{(0)}} \frac{1}{1 - z2^{-3/2}} \approx \frac{z^{(0)}}{1 - z^{(0)}2^{-3/2}}, \quad z^{(1)} \approx z^{(0)} (1 + z^{(0)}2^{-3/2}) ,$$

where $1/(1-x) \approx 1+x$ for $x \ll 1$ was used. The equation of state (13.14) for the pressure, namely $\beta P \lambda^3 \approx (2s+1)(z - z^2 2^{-5/2})$, is then

$$\begin{aligned} \beta P \lambda^3 &\approx (2s+1) [z^{(0)} (1 + z^{(0)}2^{-3/2}) - 2^{-5/2}(z^{(0)})^2] \\ &= n\lambda^3 \left(1 + 2^{-5/2} \frac{n\lambda^3}{2s+1}\right), \end{aligned}$$

when

$$\frac{1}{\sqrt{2}^3} - \frac{1}{\sqrt{2}^5} = \frac{2}{\sqrt{2}^5} - \frac{1}{\sqrt{2}^5} = \frac{1}{\sqrt{2}^5}$$

is used. Altogether we then find

$$\boxed{PV = \langle \hat{N} \rangle k_B T \left(1 + \frac{n\lambda^3}{4\sqrt{2}(2s+1)}\right)} . \quad (13.16)$$

In this expression, the first term corresponds to the equation of state for the classical ideal gas, while the second term is the first *quantum mechanical correction*.

13.3 Degenerated Fermi gas

In the *low temperature limit*, $T \rightarrow 0$, the Fermi distribution function behaves like a step function:

$$n_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1} \xrightarrow{T \rightarrow 0} \begin{cases} 0 & \text{if } \epsilon_k > \mu \\ 1 & \text{if } \epsilon_k < \mu \end{cases}$$

i.e.,

$$\lim_{T \rightarrow 0} n_k = \theta(\mu - \epsilon_k) .$$

Fermi energy. This means that all the states with energy below the Fermi energy ϵ_F ,

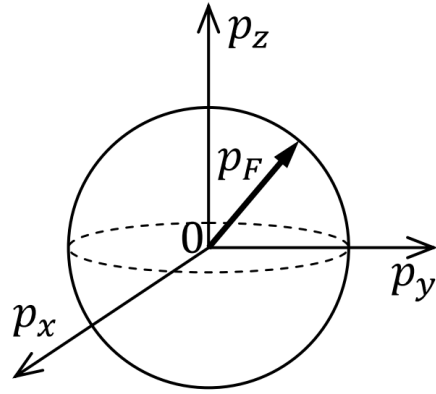
$$\epsilon_F = \mu(n, T = 0) ,$$

are occupied and all those above are empty.

Fermi sphere. In momentum space the occupied states lie within the Fermi sphere of radius p_F . The system is then deep in the quantum regime.

The Fermi energy is then determined by the condition that the Fermi sphere contains the correct number of states:

$$N = \sum_{\substack{\text{states } r \\ \text{with } \epsilon_r < \epsilon_F}} 1 ,$$



which can be written for the case of free fermions, and with (13.1), $d^3k/\Delta k^3 = [V/(2\pi)^3]d^3k$, as

$$N = \frac{(2s+1)V}{(2\pi)^3} \int_{|k| < |k_F|} d^3k = \frac{(2s+1)V}{(2\pi)^3} \frac{4}{3} \pi k_F^3 . \quad (13.17)$$

Here, $k_F = \frac{p_F}{\hbar}$ is the Fermi wave number. We have

$$n = \frac{N}{V} = \frac{2s+1}{6\pi^2} k_F^3, \quad k_F = \left(\frac{6\pi^2 n}{2s+1} \right)^{1/3} .$$

The Fermi energy is then

$$\epsilon_F = \frac{\hbar^2 k_F^2}{2m}, \quad \boxed{\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{2s+1} \right)^{2/3}} .$$

13.3.1 Ground state properties

At $T = 0$, the system is in its ground state, with the internal energy U_0 given by

$$\begin{aligned} U_0 = \sum_{|\vec{k}| < k_F} \epsilon_k &= \frac{(2s+1)V}{(2\pi)^3} \int_0^{k_F} dk (4\pi k^2) \frac{\hbar^2 k^2}{2m} \\ &= \frac{(2s+1)V}{(2\pi)^3} \left(\frac{\hbar^2}{2m} \right) \frac{4\pi}{5} k_F^5. \end{aligned}$$

Using the expression for total particle number N ,

$$N = \frac{V(2s+1)}{(2\pi)^3} \frac{4\pi}{3} k_F^3,$$

for $k_F^5 = k_F^3 k_F^2$, one obtains

$$\frac{U_0}{N} = \frac{3\hbar^2 k_F^2}{5 \cdot 2m}, \quad \boxed{\frac{U_0}{N} = \frac{3}{5} \epsilon_F} \quad (\text{independent of } s).$$

for internal energy per particle at absolute zero.

Pressure. Since $PV = 2U/3$, we obtain now an expression for the zero-point pressure P_0 :

$$\boxed{P_0 \equiv P_{T=0} = \frac{2}{5} n \epsilon_F}.$$

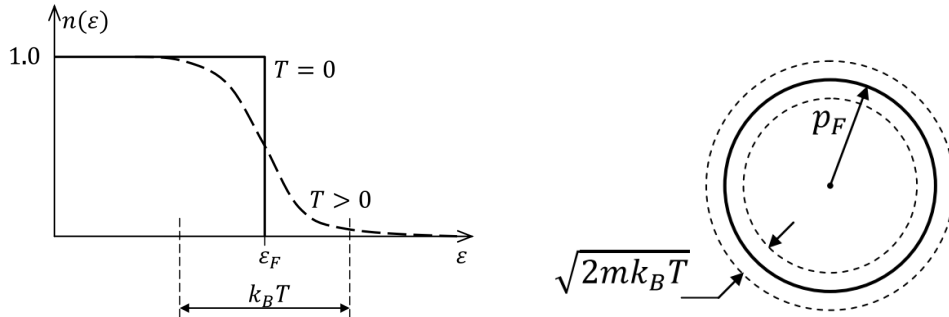
- The zero-point pressure arises from the fact that fermionic particles move even at absolute zero. This is because the zero-momentum state can hold only one particle of a given spin state.
- Taking a Fermi energy of typically $\epsilon_F \approx 10 \text{ eV} = 16 \cdot 10^{-19} \text{ J}$ and an electron density of $n \approx 10^{22} \cdot 100^3 \text{ m}^{-3}$ we find a zero-point pressure of

$$P_0 \approx 3.2 \cdot 10^3 \cdot 10^6 \frac{\text{J}}{\text{m}^3} \approx 3.2 \cdot 10^4 \text{ bar},$$

where we have used that $1 \text{ P} = 1 \text{ J/m}^3 = 10^{-5} \text{ bar}$.

13.3.2 Fermi temperature

At low but a finite temperature, the Fermi distribution function $\langle \hat{n}_r \rangle = n(\epsilon)$ for the occupation number smooths out around the Fermi energy.



Such an evolution of $n(\epsilon)$ with increasing temperature is due to the excitation of fermions within a layer beneath the Fermi surface to a layer above. “Holes” are left beneath the Fermi surface.

Fermi temperature. We define the *Fermi temperature* T_F as

$$\boxed{\epsilon_F = k_B T_F} .$$

– $T \ll T_F$

For low temperatures $T \ll T_F$, the Fermi distribution deviates from that at $T = 0$ mainly in the neighborhood of ϵ_F in a layer of thickness $k_B T$. Particles at energies of order $k_B T$ below the Fermi energy are excited to energies of order $k_B T$ above the Fermi energy.

– $T \gg T_F$

For $T \gg T_F$, the Fermi distribution approaches the Maxwell-Boltzmann distribution. The quantum nature of the constituent particles becomes irrelevant.

Frozen vs. active electrons. The typical magnitude,

$$\epsilon_F \approx 2 \text{ eV}, \quad T_F \approx 2 \times 10^4 \text{ K} ,$$

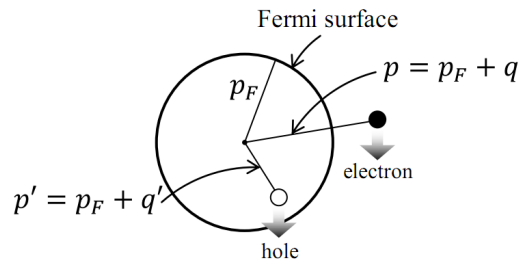
of the Fermi temperature in metals implies that room temperature electrons are frozen mostly below the Fermi level. Only a fraction of the order of

$$\frac{T}{T_F} \approx 0.015$$

of the electrons contributes to thermodynamic properties involving excited states.

Particles and holes. We can define that the absence of a fermion of energy ϵ , momentum \vec{p} and charge e corresponds to the presence of a hole with

$$\begin{aligned} \text{energy} &= -\epsilon \\ \text{momentum} &= -\vec{p} \\ \text{charge} &= -e \end{aligned}$$



The concept of a hole is useful only at low temperatures $T \ll T_F$, when there are few holes below the Fermi surface. The Fermi surface “disappears” when $T \gg T_F$, with the system approaching the Maxwell-Boltzmann distribution function.

Specific heat. Since the average excitation energy per particle is $k_B T$, the internal energy of the system is of order

$$U \approx U_0 + \left(\frac{T}{T_F} \right) N k_B T , \tag{13.18}$$

where U_0 is the ground-state energy. The specific heat capacity C_V is then of the order of

$$\boxed{\frac{C_V}{Nk_B} \sim \frac{T}{T_F}}, \quad C_V = \left(\frac{\partial U}{\partial T} \right)_V. \quad (13.19)$$

The electronic contribution to the specific heat vanishes linearly with $T \rightarrow 0$. The room-temperature contribution of phonons (lattice vibrations) to C_V is therefore in general dominant.

13.4 Low temperature expansion

In this section we will derive the scaling relations (13.18) and (13.19) together with the respective prefactors.

13.4.1 Density of states

We will work from now on with density of state per volume $D(E) = \Omega(E)/V$, which is defined as the derivative of the integrated phase space per volume, $\phi(E) = \Phi(E)/V$:

$$D(E) = \frac{\partial \phi(E)}{\partial E}, \quad \phi(E) = \frac{1}{V} \int_{\epsilon_{\mathbf{k}} \leq E} \frac{d^3 k}{\Delta k^3}, \quad \Delta k^3 = \frac{(2\pi)^3}{V}.$$

The spin degeneracy factor $2s + 1$ will be added further below.

Fermi sphere. With the phase space being isotropic we may write the volume of the Fermi sphere as

$$(V \Delta k^3) \phi(E) = \frac{4}{3} \pi k_E^3 = \frac{4\pi}{3} \left(\frac{2mE}{\hbar^2} \right)^{3/2}, \quad E = \frac{\hbar^2 k_E^2}{2m}.$$

Introducing the spin degeneracy factor $2s + 1$ we then obtain

$$\boxed{D(E) = \begin{cases} A \sqrt{E} & \text{if } E \geq 0, \\ 0 & \text{otherwise,} \end{cases}} \quad (13.20)$$

$$A = \frac{2s + 1}{(2\pi)^2} \left(\frac{2m}{\hbar^2} \right)^{3/2}$$

Since we did not use any special properties of a fermionic system, this expression is also valid for bosons. For both types of systems, $D(E)$ shows a \sqrt{E} dependence.

Energy and particle density. The energy density U/V is given by

$$\frac{U(T, \mu)}{V} = \int_{-\infty}^{+\infty} dE E n(E) D(E), \quad n(E) = \frac{1}{e^{\beta(E-\mu)} + 1}, \quad (13.21)$$

where $n(E)$ is the Fermi distribution as a function of the energy. The analogous expression for the particle density is

$$n(T, \mu) = \frac{\langle \hat{N} \rangle}{V} = \int_{-\infty}^{+\infty} dE n(E) D(E). \quad (13.22)$$

For a bosonic system one substitutes the Boson distribution function $1/(e^{\beta(E-\mu)} - 1)$ for $n(E)$.

13.4.2 Sommerfeld expansion

We are interested in the thermodynamic properties of a fermionic system at small but finite temperature, viz at a low temperature expansion of expectation values like (13.21) and (13.22).

Sommerfeld expansion. We denote with $H(E)$ a function depending exclusively on the one-particle energy E . We will show that

$$\langle H \rangle = \int_{-\infty}^{\infty} dE H(E) n(E) \approx \int_{-\infty}^{\mu} dE H(E) + \frac{\pi^2}{6} (k_B T)^2 H'(\mu) \quad (13.23)$$

holds terms of order T^4 or higher.

- The first term on the r.h.s. of (13.23) survives when $T \rightarrow 0$ and $n(E) \rightarrow \theta(\mu - E)$. It represents the ground-state expectation value.
- The second term results from expanding both $H(E)$ and $n(E)$ around the chemical potential μ .

Partial integration. We start the derivation of the Sommerfeld expansion with the definition

$$K(E) = \int_{-\infty}^E dE' H(E'), \quad H(E) = \frac{dK(E)}{dE},$$

which allows us to perform the partial integration

$$\langle H \rangle = \int_{-\infty}^{\infty} dE \frac{dK(E)}{dE} n(E) = \int_{-\infty}^{\infty} dE K(E) \left(-\frac{dn(E)}{dE} \right). \quad (13.24)$$

- For the integration we have used $\lim_{E \rightarrow \infty} n(E) = 0$, namely that the probability to find particles at elevated energies E falls off exponentially.
- We have also assumed that $\lim_{E \rightarrow -\infty} H(E) = 0$.

Taylor expansion. Substituting the first two terms of the Taylor expansion

$$K(E) = K(\mu) + (E - \mu)K'(\mu) + \frac{(E - \mu)^2}{2} K''(\mu) + O((E - \mu)^3)$$

of $K(E)$ around the chemical potential μ into (13.24) leads to

$$\langle H \rangle \approx \int_{-\infty}^{\infty} dE \left[K(\mu) + (E - \mu)K'(\mu) + \frac{(E - \mu)^2}{2}K''(\mu) \right] \left(-\frac{dn(E)}{dE} \right). \quad (13.25)$$

Ground state contribution. The first term in (13.25) is

$$K(\mu) \int_{-\infty}^{\infty} dE \left(-\frac{dn(E)}{dE} \right) = K(\mu) [n(-\infty) - n(\infty)] = \int_{-\infty}^{\mu} dE H(E),$$

viz the $T = 0$ value of $\langle H \rangle$.

Symmetry cancellation. The second term in (13.25) vanishes because

$$-\frac{dn}{dE} = \frac{\beta e^{\beta(E-\mu)}}{[e^{\beta(E-\mu)} + 1]^2} = \frac{\beta e^{-\beta(E-\mu)}}{[1 + e^{-\beta(E-\mu)}]^2}, \quad n(E) = \frac{1}{e^{\beta(E-\mu)} + 1},$$

is symmetric in $E - \mu$.

Finite temperature correction. The third term in (13.25) yields the first non-trivial correction

$$K''(\mu) \int_{-\infty}^{\infty} dE \frac{(E - \mu)^2}{2} \left(-\frac{dn(E)}{dE} \right) = \frac{\pi^2}{6} (k_B T)^2 H'(\mu),$$

where the scaling with $(k_B T)^2 = 1/\beta^2$ follows from a transformation to dimensionless variables $y = \beta(E - \mu)$. The factor $\pi^2/6$ results from the final dimensionless integral.

This concludes our derivation of the Sommerfeld expansion (13.23).

13.4.3 Internal energy at low temperatures

We start applying the Sommerfeld expansion (13.23) to the particle density (13.22):

$$\begin{aligned} n(T, \mu) &= \int_{-\infty}^{+\infty} dE D(E) n(E) \\ &= \int_{-\infty}^{\mu} dE D(E) + \frac{\pi^2}{6} (k_B T)^2 D'(\mu) \\ &= \int_{-\infty}^{\epsilon_F} dE D(E) + \underbrace{\left\{ \int_{\epsilon_F}^{\mu} dE D(E) + \frac{\pi^2}{6} (k_B T)^2 D'(\mu) \right\}}_{\approx (\mu - \epsilon_F) D(\epsilon_F)}, \end{aligned} \quad (13.26)$$

where we have taken into account that $\mu = \mu(T)$ may be different from the Fermi energy $\epsilon_F = \lim_{T \rightarrow 0} \mu(T)$. We have also assumed that the density of states $D(E)$ is essentially constant around the Fermi energy.

Constant particle density. The terms inside the bracket in (13.26) need to cancel if the particle density n is to be constant. The chemical potential μ varies hence as

$$\mu = \epsilon_F - \frac{\pi^2}{6} (k_B T)^2 \frac{D'(\epsilon_F)}{D(\epsilon_F)}, \quad \boxed{\mu = \epsilon_F - \frac{\pi^2 (k_B T)^2}{12 \epsilon_F}} \quad (13.27)$$

as a function of temperature.

- We have approximated $D'(\mu)$ in (13.27) consistently by $D'(\epsilon_F)$.
- The scaling $D(E) \sim \sqrt{E}$, as given by (13.20), leads to $D'/D = 1/(2E)$.

Internal energy. The energy density (13.21) evaluated with the Sommerfeld expansion is

$$\begin{aligned} u = U/V &= \int_{-\infty}^{+\infty} dE E n(E) D(E) \\ &= u_0 + \int_{\epsilon_F}^{\mu} dE E D(E) + \frac{\pi^2}{6} (k_B T)^2 \underbrace{\frac{d}{dE} (ED(E))}_{\frac{3}{2} D(\epsilon_F)} \Big|_{E=\mu}, \end{aligned}$$

where $u_0 = \int_{-\infty}^{\epsilon_0} ED(E)dE$ is the ground state energy. The substitution $\mu \rightarrow \epsilon_F$ performed for the argument of last term, for which we used $D(E) \sim \sqrt{E}$, is correct to order T^2 . The second term is

$$\int_{\epsilon_F}^{\mu} dE E D(E) \approx (\mu - \epsilon_F) \epsilon_F D(\epsilon_F) = -\frac{\pi^2 (k_B T)^2}{12} D(\epsilon_F).$$

when $D(E) \sim \sqrt{E}$. One finds hence with $3/12 - 1/12 = 1/6$ that

$$\frac{U}{V} = u_0 + \frac{\pi^2 (k_B T)^2}{6} D(\epsilon_F) \quad \boxed{u - u_0 \propto (k_B T)^2 D_0}. \quad (13.28)$$

The internal energy increases quadratically with the temperature, being at the same time proportional to the density of states $D_0 = D(\epsilon_0)$ at the Fermi level.

Specific heat. Assuming a constant density of states $D(E) \approx D(\epsilon_F)$ close to the Fermi energy ϵ_F we find

$$c_V = \frac{C_V}{V} = \frac{1}{V} \frac{\partial U}{\partial T}, \quad \boxed{c_V \approx \frac{\pi^2}{3} (k_B^2 T) D(\epsilon_F)} \quad (13.29)$$

for the intensive specific heat. A trademark of a fermionic gas is that c_V is linear in the temperature.

The heat capacity per volume saturates however for $T \rightarrow \infty$, where it becomes identical with the ideal gas value $C_V/V = 3nk_B/2$ derived in Sec. 3.5.1.

