Exercise Sheet #4

Oren Neumann <neumann@itp.uni-frankfurt.de>
Daniel Nevermann <nevermann@itp.uni-frankfurt.de>

Problem 1 (*Time-Delayed Differential Equation, 10pts*)

Consider a car following a truck. The driver of the car should accelerate or break depending on the velocity of the truck that is in front. As the driver has a certain non-zero reaction time $T > 0$, the acceleration of the car is given by

$$\dot{v}(t+T) = \alpha \left(v_{\text{truck}}(t) - v(t) \right) \ ,$$

where $v(t)$ is the velocity of the car, v_{truck} is the velocity of the truck and $\alpha > 0$ is an acceleration parameter of the car.

Assume that the truck has a constant speed of $v(t) \equiv v_0$. Analyze the solution for the velocity $\dot{x}(t)$ for this case. Study the stability of this solution as function of the parameters α and T.

Problem 2 (*Discrete SIR Model, 10pts*)

In the lecture, the discrete SIR model was given as

$$x(t+1) = ax(t)\left[1 - \sum_{k=0}^{\tau_{\text{R}}} x(t-k)\right] \ .$$

Using $\tau_{\text{R}} = 0$ gives you the logistic map, which undergoes a Hopf bifurcation at $a = 3$. We would like to find the value of a for this critical point for different values of τ_{R}. The following hints should guide you to a solution that expresses the critical value for a (that is, the value where the fixed point loses its stability) as a function of τ_{R}.

a) Find an expression for the non-trivial fixed point of the system for given a and τ_{R}.

b) Transform the system into a coordinate frame that puts the fixed point to zero.

c) Use the exponential "ansatz" $x(t) = x_0 b^t$, keeping in mind that generally, $b \in \mathbb{C}$.

d) State the proper condition on b corresponding to the transition point between a stable/unstable fixed point.

e) You should get a set of equations, where the following identities are useful to proceed:
$\sum_{k=0}^{n} \sin(xk) = \sin(nx/2)\sin((n+1)x/2)/\sin(x/2)$,
$\sum_{k=0}^{n} \cos(xk) = \cos(nx/2)\sin((n+1)x/2)/\sin(x/2)$.

f) You should eventually arrive at an explicit expression for the critical parameter $a_{\text{c}}(\tau_{\text{R}})$ as a function of τ_{R}. It is also ok to use algebraic math tools like wolframalpha.com to lead you to the solution. Sketch this function. What is the effect of larger τ_{R}?