Exercise Sheet #3

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Problem 1 (*Period–3 cycle, 10pt*)

In chaotic dynamics that arise from a cascade of period doubling bifurcations, one always finds windows of regular motion with respect to the bifurcation parameter. The widest of such windows shows a period–3 cycle. Consider the logistic map as an example system to study the occurrence of period–3 cycles:

$$x_{n+1} = f(x_n) \tag{1}$$
$$f(x_n) = r x_n (1 - x_n)\,, \tag{2}$$

where the variable $0 \le x_n \le 1$ and the bifurcation parameter $0 \le r \le 4$.

a) In order to exert a period–3 cycle, the third iterated map has to fulfill the fixpoint condition $f(f(f(x_n))) = x_n$, while the map itself does not $f(x_n) \neq x_n$. Plot the third iterated for different values of the bifurcation parameter and try to understand how the period–3 cycle evolves.

b) Calculate the lower bound r_3 for the bifurcation parameter, for which a period–3 cycle can exist. Follow these steps:

   i) Consider the three points of the cycle $x_n \to x_{n+1} \to x_{n+2} \to x_n$ being ordered and substitute $x = x_n$, $y = x_{n+1}$, $z = x_{n+2}$. Then you should find three equations linking x, y, z.

   ii) As a fourth condition you can use the fact that the derivative of the third iterated function $\mathrm{d}f(f(f(x)))/\mathrm{d}x = 1$ is unity as it touches the diagonal. Re–write that condition in terms of x, y, z using the chain rule.

   iii) Re–write the four conditions found above by the following substitution:

$$A = r\left(x - \frac{1}{2}\right), \qquad B = r\left(y - \frac{1}{2}\right), \qquad C = r\left(z - \frac{1}{2}\right). \tag{3}$$

   You should find that the equations are invariant under the cycle permutation of $A \to B \to C \to A$.

   iv) As the solution is invariant under cyclic permution, each of the variables x, y, z has to fulfill the equation:

$$(x - A)(x - B)(x - C) = 0\,. \tag{4}$$

   Use Eq. (4) to find the solution for A, B, C and finally the value of r_3. Keep in mind that you have to exclude the case $x = y = z$.

c) At the point $r = r_3$ the period–3 cycle comes into existence, but one has to show that for $r = r_3 + \Delta r$ the cylce is stable for small but finite $\Delta r > 0$. Therefore approximate one of the tangent points $x(t)$ of the third iterated map (where it touches the diagonal $x_{n+1} = x_n$) by a quadratic function that is shifted down (up) by $\Delta r > 0$ ($\Delta r < 0$) touching at $\Delta r = 0$. Show that one of the intersections with the diagonal has a slope less than one (absolute value) for some range in $\Delta r > 0$.

Problem 2 (*Triangle Map, 10pt*)

Consider the tent map:

$$f(x) = \begin{cases} rx & \text{if } 0 \le x < 1/2 \\ r(1-x) & \text{if } 1/2 \le x \le 1 \end{cases} , \tag{5}$$

for $0 \le x \le 1$ and the parameter $0 \le r \le 2$.

a) Plot the function.

b) Look for fixed points and cycles (up to length 3).

c) Derive the analytic expression for the maximal Lyapunov exponent, defined by

$$\lambda_{\text{max}} = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\mathrm{d}f^{(n)}}{\mathrm{d}x} \right| , \text{ where } f^{(n)}(x) = f\left(f^{(n-1)}(x)\right) . \tag{6}$$

Hint: use the chain rule.

d) For which range of r does the triangle map exhibit chaos?