Exercise Sheet #11

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Problem 1 (*Landau-Ginzburg Theory, 7pts*)

Consider a system with the following free energy density:

$$f(\phi, T) = \frac{1}{2}\gamma(T^2 - T_0^2)\phi^2 - \frac{1}{3}\alpha T\phi^3 + \frac{1}{4}\lambda\phi^4$$

where α, γ, λ and T_0 are positive parameters.

a) Show that the system has a first order phase transition at the critical temperature $T_c^2 = T_0^2/[1 - \frac{2\alpha^2}{9\lambda\gamma}]$

b) Show that the second minimum at $\phi > 0$ exists only when $T^2 \leq 8T_0^2/[9(\frac{T_0}{T_c})^2 - 1]$

Problem 2 (*Probability Generating Functions, 7pts*)

Generating functions can be very useful for sums in processes with multiple random variables. To show this we will look at a dice game, where a player moves X steps on a board each turn according to the number rolled on the dice.

a) Write down the generating function $G_X(z)$ of the probability to get the number X on a six-sided dice throw.

b) During a game the player advances each turn according to a dice throw. The number of turns is N and the total sum of steps taken during the game is S_N. Derive the generating function $G_{S_N}(z)$ of S_N as a function of $G_X(z)$.

c) Now assume N is a random variable with distribution p_N, which is generated by the function $G_N(z)$. Prove that $G_{S_N}(z)$ is now given by:

$$G_{S_N}(z) = G_N(G_X(z))$$

Hint: Use the law of total expectation, $E(x) = E_y(E(x|y))$.

d) Assuming that N is a poissonian variable with probability $p_N = e^{-\lambda}\frac{\lambda^N}{N!}$, calculate the average number of steps in a game $\langle S_N \rangle$ using your previous result.

Problem 3 (*Galton Watson Process, 6pts*)

You have seen in the lecture notes (Eq. 5.32) the stationary condition for extinction of a family name in the Galton-Watson process:

$$q = G^{(n)}(0) = G_0(G^{(n-1)}(0)) = G_0(q)$$

Where $\lim_{n\to\infty} G^{(n)}(0) = q$ is the extinction probability at a late generation n. Use this condition to prove that the extinction probability is 1 when the average reproduction rate is smaller than 1.

Hint: It may help to look at the function $F(x) \doteq G_0(x) - x$.