Exercise Sheet #1

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This Exercise sheet is a warm-up, it does not count towards your final grade and there is no need to hand it in.

Problem 1 (*Harmonic Oscillator, 10pts*)

The dynamics of the classical harmonic oscillator is determined by the second order ordinary differential equation (ODE)

$$\ddot{x}(t) = -\omega_0^2 x(t) - \alpha \dot{x}(t), \tag{1}$$

where $\omega_0 > 0$ is the natural frequency of the oscillator and α is the damping factor. Some prominent cases of operating regimes for this system are:

- Strong damping: $\frac{\alpha}{2} > \omega_0$.
- Weak damping: $0 < \frac{\alpha}{2} < \omega_0$.
- No damping: $\alpha = 0$.
- Negative damping, i.e. energy uptake: $\alpha < 0$.

a) Solve Eq. 1 for the initial conditions $x(t_0) = x_0, \dot{x}(t_0) = v_0$ analytically. Then sketch the system's long time evolution, i.e. $x(t)$ for $t \gg t_0$, for each of the four cases mentioned above.

b) Now investigate the harmonic oscillator as a dynamical system.

- Re-write the second order ODE (Eq. 1) as a system of first order ODEs in x, y where $y = \dot{x}$.
- Find the fixed point of the re-written ODE.
- Sketch the flow of the system in the phase space for each of the four cases mentioned above and indicate the fixed point in the sketches. For which case is it (un)stable?

c) Compare the analytic solution of Eq. 1 you calculated in (a) with the dynamical systems' point of view obtained in (b).

Problem 2 (*Topological Equivalence, 10 pts*)

Two maps

$$x(t+1) = f(x(t)) \tag{2}$$
$$y(t+1) = g(y(t)) \tag{3}$$

are said to be topologically equivalent if there exists a continuous invertible function $y = h(x)$ such that (2) is mapped onto (3), that is

$$h(f(x)) = g(h(x)) \quad , \forall x. \tag{4}$$

This also means that there is an invertible, continuous mapping between the solutions of both maps. Now, consider the two maps

$$x(t+1) = ax(t) \tag{5}$$
$$y(t+1) = by(t) \tag{6}$$

where $a, b \in \mathbb{R}$. Try to find an appropriate function $h(x)$ that fulfills (4) for (5) and (6) (hint: which type of function is 'scale invariant'?). For which choices of a and b is this possible? Sketch some examples of sequences of $x(t)$ and $y(t)$ for which the maps are topologically equivalent/not equivalent. How do you interpret these results?