Goethe-Universität Frankfurt
Institut für Theoretische Physik

Lecturer: Prof. Dr. Claudius Gros, Room 1.132
Tutorial supervisor: Dr. Francesco Ferrari, Room 1.143

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Höhere Quantenmechanik

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Exercise sheet 8

(Submission date: Until 13.06.2022 12:00)

Exercise 1: squeezed coherent states *(14 Points)*

In the context of quantum optics, the so-called *squeezing operator* is defined as

$$S(\eta) = e^{\frac{1}{2}(\eta(a^\dagger)^2 - \eta^* a^2)},$$

where a and $a^\dagger$ are ladder operators (or, in other words, bosonic annihilation and creation operators). $\eta \in \mathbb{C}$ is a complex number.

(i) Show that $S(\eta)$ is a unitary operator that satisfies $S^{-1}(\eta) = S^\dagger(\eta) = S(-\eta)$. *(1 Point)*

Hint: you may want to use Baker–Campbell–Hausdorff formula

$$e^X e^Y = e^Z, \text{ where } Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + ...$$

(ii) Prove that

$$S^\dagger(\eta) a S(\eta) = \cosh(r) a + e^{i\theta} \sinh(r) a^\dagger, \tag{1}$$

where r and θ are two real numbers such that $\eta = re^{i\theta}$. You can use

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots \tag{2}$$

and carry out the proof up to the third order of the expansion. Then just comment about what happens for higher orders. *(4 Points)*

(iii) We consider a coherent state $|c\rangle = D(c)|0\rangle$, $c \in \mathbb{C}$ (see exercise sheet 6). Show that it satisfies $\Delta x \Delta p = \frac{\hbar}{2}$ (i.e., Heisenberg's uncertainty principle with the equal sign), where $x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$ and $p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a)$ are the position and momentum operators, respectively. *(3 Points)*

(iv) By making use of $S(\eta)$, one can define a so-called *squeezed coherent state*, as follows

$$|c, \eta\rangle = D(c) S(\eta) |0\rangle \tag{3}$$

Verify that $\Delta x \Delta p = \frac{\hbar}{2}$ is true also for squeezed coherent states. For simplicity, you can set $\theta = 0$, such that η becomes real ($\eta = r \in \mathbb{R}$). Discuss the difference with the previous case of the simple coherent state: why is $|c, \eta\rangle$ a "squeezed" state? what happens if $r \to \pm\infty$?. *(6 Points)*

Exercise 2: creation and annihilation operators, fermions and bosons ***(6 Points)***

(i) For a system of identical fermions, compute $\langle 0|c_\alpha c_\beta c_\alpha^\dagger c_\beta^\dagger|0\rangle$ (for $\alpha \neq \beta$) *(1 Point)*

(ii) For a system of identical bosons, compute $\langle 0|b_\alpha b_\beta b_\alpha^\dagger b_\beta^\dagger|0\rangle$ (for $\alpha \neq \beta$) *(1 Point)*

(iii) For a system of identical fermions, compute $\langle 0|c_1 c_3 c_4 c_2^\dagger c_2 c_5^\dagger c_3^\dagger c_5 c_4 c_4^\dagger c_1 c_5^\dagger c_1^\dagger c_2^\dagger|0\rangle$ *(1 Point)*

(iv) Show that, for both bosons and fermions, the number operators commute, i.e. $[n_\alpha, n_\beta] = 0$. *(3 Points)*