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Höhere Quantenmechanik

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Exercise sheet 7

(Submission date: Until 06.06.2022 12:00)

Exercise 1: two-level system subjected to a radiation *(12 Points)*

We consider a two-level system (analogous to a spin-$\frac{1}{2}$ degree of freedom) in presence of an electromagnetic radiation, described by the Hamiltonian

$$H = H_0 + H_1 \tag{1}$$

$$H_0 = -\frac{1}{2}\Omega\sigma_z + \sum_{\vec{k}} \hbar\omega_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} \tag{2}$$

$$H_1 = \sum_{\vec{k}} \hbar g_{\vec{k}} (a_{\vec{k}}\sigma_+ + a^\dagger_{\vec{k}}\sigma_-), \tag{3}$$

where $\sigma_\pm = \sigma_x \pm i\sigma_y$.

(i) What is the physical meaning of the Hamiltonian? *(1 Point)*

(ii) Write H_1 in the interaction picture. You should obtain an expression of the form

$$H_1'(t) = \sum_{\vec{k}} (O_{\vec{k}} e^{-i\Omega t} + O^\dagger_{\vec{k}} e^{i\Omega t})$$

What is the operator $O_{\vec{k}}$? *(5 Points)*

(iii) Let us denote by g and e the ground state and excited state of the two-level system, respectively. Compute the transition rates $\Gamma_{(g,n)\to(e,n')}$ and $\Gamma_{(e,n)\to(g,n')}$, where $n = \{n_{\vec{k}}\}$ and $n' = \{n'_{\vec{k}}\}$ represent the set of initial and final photon numbers. Discuss the results and their physical meaning. *(6 Points)*

Exercise 2: paramagnetism and diamagnetism *(8 Points)*

We consider the Hamiltonian of a hydrogen atom in a uniform magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 - \frac{e}{m}\vec{B}\cdot\vec{S} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} \tag{4}$$

Without loss of generality, we can assume that $\vec{B}$ is oriented along the z-axis. Note: we use SI units in this exercise (for this reason there is no speed of light c in the minimal coupling expression).

(i) Show that the symmetric gauge $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$ is a valid choice for the vector potential $\vec{A}$. Is it a Coulomb gauge? *(1 Point)*

(ii) Use the symmetric gauge expression for $\vec{A}$ in the Hamiltonian and identify the contributions due to B. In particular, you should recognize the *paramagnetic* term (involving angular momentum and spin of the electron) and the *diamagnetic* term (quadratic in B). *(2 Points)*

(iii) The effect of the diamagnetic term on the ground state of the hydrogen atom is negligible when compared to the paramagnetic contribution. Can you show why? Consider a magnetic field of strength $B = 1\ T$. *(3 Points)*

Hint: for the ground state of the hydrogen atom, $\langle \Psi_0 | r^2 | \Psi_0 \rangle = 3a_0^2$ *(*a_0*=Bohr radius).*

(iv) For which chemical elements (in the form of a single atom) can we expect the diamagnetic contribution to become more relevant? Provide an example. *(2 Points)*