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Höhere Quantenmechanik

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Exercise sheet 6

(Submission date: Until 30.05.2022 12:00)

Exercise 1: harmonic oscillator in a magnetic field *(7 Points)*

We consider a charged and anisotropic two-dimensional harmonic oscillator described by the Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2) \tag{1}$$

At $t = 0$, the system is in a stationary state $|\psi_0\rangle = |n_1\rangle \otimes |n_2\rangle = |n_1, n_2\rangle$ (n_1 and n_2 are the occupation numbers for the x- and y-components of the oscillator, respectively). An external magnetic field is switched on $t = 0$, for a certain time interval T. The effect of the field can be approximately described by the perturbation $V = -\frac{q}{2mc} B L_z$, where q is the charge of the oscillator, c is the speed of light and L_z is the z component of the angular momentum operator. Note: the field strength B does not depend on time.

Using first-order perturbation theory, compute the probability that, at time $t > T$, the system is in the stationary state $|\psi_f\rangle = |n_2, n_1\rangle$, with the quantum numbers having been swapped by the perturbation. For what values of n_1 and n_2 can this process take place? Discuss also what happens in the special case $\omega_x = \omega_y$. *(7 Points)*

Exercise 2: coherent states of harmonic oscillator *(10 Points)*

A coherent state $|c\rangle$ can be written as

$$|c\rangle = D(c)|0\rangle, \tag{2}$$

where $|0\rangle$ is the vacuum state and $D(c) = e^{ca^\dagger - c^* a}$, with a and $a^\dagger$ being ladder operators of a harmonic oscillator, and c a complex number.

(i) Use Baker-Campbell-Hausdorff formula (X, Y are two generic operators)

$$e^X e^Y = e^Z, \text{ where } Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + ...$$

to show that $D(c)D(z)$ is proportional to $D(c+z)$ through a phase factor. *(1 Point)*

(ii) $D(c)$ is called *displacement operator*. To see why, compute $D^{-1}(c) a D(c)$. *(1 Point)*

Hint: use the identity

$$e^A B e^{-A} = B + [A,B] + \frac{[A,[A,B]]}{2!} + ...$$

(iii) Show that the displacement operator can be equivalently written as $D(c) = e^{-\frac{|c|^2}{2}} e^{ca^\dagger} e^{-c^* a}$. *(1 Point)*

(iv) Write the state $|c\rangle$ as a Fock expansion over states labelled by the occupation number, i.e. $|c\rangle = \sum_n \lambda_n |n\rangle$. What are the coefficients λ_n? *(1 Point)*

(v) Show that different coherent states are not orthogonal to each other. Show also that coherent states are normalized. *(1 Point)*

(vi) Show that one can write the identity operator as $\mathbb{1} = \int_{\mathbb{C}} \frac{dc}{\pi} |c\rangle\langle c|$. *(3 Points)*

Hint: assume that integrals and sums can be swapped without problems. Remember that, for an integer number n, we have

$$\int_0^\infty dx\ x^n e^{-x} = n!$$

(vii) Consider the special case of a coherent state $|c\rangle$ in which $c \in \mathbb{R}$ is a real number. What is the wave function $\psi_c(x) = \langle x|c\rangle$ of this state? *(2 Points)*

Exercise 3: phase operator *(3 Points)*

The phase operator $e^{i\phi}$ is discussed in the notes (5.2).

(i) What are its eigenstates? Is it possible to normalize them? *(2 Points)*

Hint: express the states in the basis of occupation numbers $\{|n\rangle\}$ (Fock expansion).

(ii) Show explicitly that the commutator $[e^{i\phi}, e^{-i\phi^\dagger}]$ is the projector onto the vacuum state. *(1 Point)*