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# Höhere Quantenmechanik

Summer term 2022

## Exercise sheet 10

(Submission date: Until 27.06.2022 12:00)

### Exercise 1: relativistic particle hitting a barrier (Klein paradox) *(14 Points)*

Consider an incoming relativistic particle with momentum $k$ (oriented along the $z$-axis) and energy $E > mc^2 > 0$,

$$\psi_{in}(z) = e^{i(kz-Et/\hbar)}v_{in} = e^{i(kz-Et/\hbar)} \begin{pmatrix} 1 \\ 0 \\ \frac{\hbar kc}{E+mc^2} \\ 0 \end{pmatrix} \tag{1}$$

It will scatter on a potential step barrier

$$V(z) = \begin{cases} V_0 & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases} \tag{2}$$

with $V_0 > 0$.

(i) Verify that $\psi_{in}$ is a solution of the Dirac equation if $V_0 = 0$ with $E^2 = c^2\hbar^2k^2 + m^2c^4$. *(4 Points)*

(ii) For $V_0 \neq 0$ consider the Ansatz

$$\psi(z) = \begin{cases} \psi_{in}(z) + \psi_r(z) & \text{for } z < 0 \\ \psi_t(z) & \text{for } z \geq 0 \end{cases}, \tag{3}$$

where $\psi_r = e^{i(-kz-Et/\hbar)}v_r$ is the reflected wave and $\psi_t = e^{i(qz-Et/\hbar)}v_t$ is the transmitted wave with

$$(E - V_0)^2 = c^2\hbar^2q^2 + m^2c^4.$$

Solve the Dirac equation by finding suitable $v_r$ and $v_t$. *(6 Points)*

*Hints: to perform a general solution, assume that $v_r$ is a linear combination of the two four-component amplitudes of Eq.(7.39) of the lecture notes (with $p_x = 0, p_y = 0$ and $p_z = -\hbar k$, since the reflected wave moves backward along the $z$ direction). For $v_t$ make a similar assumption, but replace $-k$ with $q$ and $E$ with $E - V_0$. Determine the coefficients of the superpositions by requiring $\psi(z)$ to be continuous at $z = 0$.*

(iii) Does the spin change upon reflection or transmission? *(1 Point)*

(iv) Discuss how the amplitude of the transmitted wave $\psi_t(z)$ behaves with $z$ in two cases:

a) $mc^2 > |E - V_0|$.

b) $mc^2 < |E - V_0|$.

*(3 Points)*

**Exercise 2: Bogoliubov transformations** *(6 Points)*

Consider the bosonic annihilation and creation operators $a$ and $a^\dagger$. We introduce the following transformation

$$\begin{pmatrix} b^\dagger \\ b \end{pmatrix} = \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) \\ -\sinh(\phi) & \cosh(\phi) \end{pmatrix} \begin{pmatrix} a^\dagger \\ a \end{pmatrix} \tag{4}$$

which defines a new pair of operators, $b$ and $b^\dagger$.

(i) Show that the new operators $b$ and $b^\dagger$ are still well-defined operators for bosonic particles, i.e. they satisfy the bosonic commutation relations. *(2 Points)*

(ii) The vacuum state of $b$-bosons ($|0\rangle_b$) differs from the one of $a$-bosons ($|0\rangle_a$). The former can be written in terms of the latter as $|0\rangle_b = N \exp\left[\frac{1}{2}\tanh(\phi)(a^\dagger)^2\right]|0\rangle_a$, where $N$ is a normalization constant (not important for the exercise). Show that $b|0_b\rangle = 0$. *(4 Points)*

*Hint: you can expand the exponential and you will see that the various terms cancel. If you want, before performing the general proof, you can convince yourself that the expansion works by checking what happens at the lowest orders.*