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Höhere Quantenmechanik

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Exercise sheet 1

(Submission date: Until 25.04.2022 12:00)

Exercise 1: operators and states *(6 Points)*

We consider the quantum state $|\psi\rangle$ which is a simultaneous eigenstate of the operators $\mathbf{A}$ and $\mathbf{B}$ with eigenvalues a and b, respectively. For simplicity, we also assume that $\mathbf{A}$ and $\mathbf{B}$ have a discrete spectrum.

(i) if the two operators *anticommute*, i.e. $\mathbf{AB} + \mathbf{BA} = 0$, what can you say about the eigenvalues a and b? *(1 Point)*

(ii) Denoting by $|x\rangle$ the eigenstates of the position operator ($x \in \mathbb{R}$), we can define the parity operator $\mathbf{P}$ such that $\mathbf{P}|x\rangle = |-x\rangle$. What is the effect of the operator $\frac{1}{2}(\mathbb{1} + \mathbf{P})$ ($\mathbb{1}$ being the identity) on a generic state $|\psi\rangle$? *(1 Point)*

(iii) Prove that $\mathbf{P}$ is self-adjoint and that $\mathbf{P}^2 = \mathbb{1}$. What are the possible eigenvalues of $\mathbf{P}$? *(2 Points)*

(iv) The parity operator $\mathbf{P}$ anticommutes with the momentum operator $\mathbf{p} = -i\hbar\frac{d}{dx}$. Can you prove it? What is the *physical* implication for a quantum state that is an eigenstate of both operators? *(2 Points)*

Exercise 2: perturbing a quantum harmonic oscillator *(8 Points)*

We consider the one-dimensional quantum harmonic oscillator

$$\mathbf{H}_0 = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{x}^2$$

where m is a mass and ω a frequency.

(i) Explicitly rewrite the Hamiltonian in terms of the ladder operators $a = \sqrt{\frac{m\omega}{2\hbar}}(\mathbf{x} + \frac{i}{m\omega}\mathbf{p})$ and $a^\dagger$ (remember: $[a, a^\dagger] = 1$). What is the ground state of the system? *(1 Point)*

(ii) We want to see the effect of three different perturbations on the harmonic oscillator: $\mathbf{V}_1 = \alpha$, $\mathbf{V}_2 = \beta\mathbf{x}$, and $\mathbf{V}_3 = \gamma\mathbf{x}^2$. Considering each of these perturbation separately ($\mathbf{H} = \mathbf{H}_0 + \mathbf{V}_i$, $i = 1, 2, 3$), use time-independent perturbation theory to compute the first- and second-order corrections to the ground state energy. *(3 Points)*

(iii) Compare the results that you have obtained by perturbation theory with the exact solutions of the three cases (show explicitly how you obtain the exact ground state energies). *(4 Points)*

Exercise 3: angular momentum ***(6 Points)***

We consider the angular momentum operator $\vec{\mathbf{L}} = (\mathbf{L}_x, \mathbf{L}_y, \mathbf{L}_z)$ and the eigenstates of $\mathbf{L}_z$ and $\mathbf{L}^2$ operators, defined by

$$\mathbf{L}_z|\Psi_{l,m}\rangle = \hbar m|\Psi_{l,m}\rangle \quad \text{and} \quad \mathbf{L}^2|\Psi_{l,m}\rangle = \hbar^2 l(l+1)|\Psi_{l,m}\rangle.$$

We denote by $\mathbf{L}_+ = \mathbf{L}_x + i\mathbf{L}_y$ and $\mathbf{L}_- = \mathbf{L}_x - i\mathbf{L}_y$ the raising and lowering operators, respectively. Using the commutation relation $[\mathbf{L}_a, \mathbf{L}_b] = i\hbar\epsilon_{abc}\mathbf{L}_c$

(i) compute the results of the commutators $[\mathbf{L}_z, \mathbf{L}_\pm]$ and $[\mathbf{L}^2, \mathbf{L}_\pm]$. *(2 Points)*

(ii) Show that

$$\mathbf{L}_z\mathbf{L}_\pm|\Psi_{l,m}\rangle = \hbar(m \pm 1)\mathbf{L}_\pm|\Psi_{l,m}\rangle,$$

i.e. that $\mathbf{L}_\pm$ raise and lower the angular momentum quantum number m. Why is $\mathbf{L}_\pm|\Psi_{l,m}\rangle$ also an eigenstate of $\mathbf{L}^2$? *(2 Points)*

(iii) Show that the value of m is bounded and it can only take values $-l \leq m \leq l$ (with $l > 0$). *Hint: you can use the condition* $\|\mathbf{L}_\pm|\Psi_{l,m}\rangle\|^2 > 0$ *as a starting point (2 Points)*