Goethe-Universität Frankfurt
Institut für Theoretische Physik

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Frankfurt, 11.04.2022

Höhere Quantenmechanik

Summer term 2022

Exercise sheet 0

warm up sheet - no points assigned

(Submission date: Until 18.04.2022 12:00)

Exercise 1: sequence in Hilbert space

We consider an infinite Hilbert space $\mathcal{H}$. Given a certain (normalized) quantum state $|\psi\rangle \in \mathcal{H}$, and an orthonormal basis $\{|n\rangle\}_{n\in\mathbb{N}} \in \mathcal{H}$, show that the sequence

$$|\psi_N\rangle = \sum_{n=0}^{N} \langle n|\psi\rangle |n\rangle \tag{1}$$

converges *in norm* to $|\psi\rangle$ when $N \to \infty$, i.e. $\|\psi - \psi_N\| \xrightarrow{N\to\infty} 0$.

Exercise 2: change of basis

Consider the operator $\mathbf{U} = \sum_n |\psi'_n\rangle \langle\psi_n|$ which performs the transformation from the orthonormal basis $\{|\psi_n\rangle\}_{n\in\mathbb{N}}$ to the orthonormal basis $\{|\psi'_n\rangle\}_{n\in\mathbb{N}}$.

(i) Show that $\mathbf{U}$ is unitary.

(ii) Explicitly show that the trace of a generic operator is invariant under the basis transformation, i.e. $\mathrm{Tr}(\mathbf{A}') = \mathrm{Tr}(\mathbf{A})$.

(iii) Assuming that $\mathbf{A}|\psi\rangle = a|\psi\rangle$, show that $\mathbf{A}'|\psi'\rangle = a|\psi'\rangle$.

Exercise 3: spectral representation

Consider the Hermitian operator

$$\mathbf{A} = \begin{pmatrix} 1 & 2i & 0 \\ -2i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{2}$$

Explicitly write down its spectral representation, $\mathbf{A} = \sum_j a_j \mathbf{P}_j$, where a_j are the eigenvalues of A and $\mathbf{P}_j$ the corresponding projectors.

Exercise 4: density matrix, mixed and pure states

We consider the following orthonormal basis for the Hilbert space of a spin-1 particle:

$$|-\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |+\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3}$$

(i) Write the density matrix of the pure quantum state $|\Psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{2}|-\rangle + \frac{1}{2}|+\rangle$.

(ii) Write the density matrix of a mixed state in which the probabilities of the three states are $w_- = \frac{1}{4}$, $w_0 = \frac{1}{2}$, $w_+ = \frac{1}{4}$.

(iii) Using the density matrices of the previous two points, compute the expectation value of the operator

$$\mathbf{S}_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

(iv) Using the density matrices of the previous two points, compute the expectation value of the operator

$$\mathbf{S}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5}$$

(v) One way to measure the level of "mixedness" of a state is computing the von Neumann entropy $S = -\operatorname{Tr}[\boldsymbol{\rho} \ln(\boldsymbol{\rho})]$ (where ln is the natural logarithm and $\boldsymbol{\rho}$ the density matrix of the state). Compute the von Neumann entropy for the pure and mixed states of points (i) and (ii). Can you write down the density matrix for the *maximally mixed state*, namely the mixed state that gives the largest possible entropy for the Hilbert space of the problem ($S = \ln(3)$ in our case)?