# Teil VIII

# Mathematischer Anhang

# Kapitel 20

# Formelsammlung

## 20.1 Vektorprodukte

| | | | |
|---|---|---|---|
| Vektor: | $\mathbf{a}$ | $=$ | $a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z = (a_x, a_y, a_z)$ |
| Skalarprodukt: | $\mathbf{a}\cdot\mathbf{b}$ | $=$ | $a_xb_x + a_yb_y + a_zb_z$ |
| Vektorprodukt: | $\mathbf{a}\times\mathbf{b}$ | $=$ | $(a_yb_z - a_zb_y, a_zb_x - a_xb_z, a_xb_y - a_yb_x)$ |
| | $(\mathbf{a}\times\mathbf{b})_i$ | $=$ | $\epsilon_{ijk}a_jb_k$ |
| | | | $\epsilon_{ijk}$ : total antisymmetrischer Tensor |
| | $\epsilon_{ijk}$ | $=$ | $-\epsilon_{jik} = -\epsilon_{ikj}$ |
| | $\epsilon_{xyz}$ | $=$ | $\epsilon_{yzx} = \epsilon_{zxy} = +1$ |
| | $\epsilon_{yxz}$ | $=$ | $\epsilon_{xzy} = \epsilon_{zyx} = -1,$ 0 sonst |

| | | | |
|---|---|---|---|
| Eigenschaften: | $\mathbf{a}\times\mathbf{b}$ | $=$ | $-\mathbf{b}\times\mathbf{a} \quad \Longrightarrow \quad \mathbf{a}\times\mathbf{a} = 0$ |
| | $(\mathbf{a}+\mathbf{b})\times\mathbf{c}$ | $=$ | $\mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}, \quad (\gamma\mathbf{a})\times\mathbf{b} = \gamma(\mathbf{a}\times\mathbf{b})$ (bi-linear) |
| | $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ | $=$ | $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ |
| | $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ | $=$ | $(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ |
| | $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ | $=$ | $(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$ |
| | $\mathbf{e}_x\times\mathbf{e}_y$ | $=$ | $\mathbf{e}_z$ & und zyklisch permutiert |

## 20.2 Differentiation in 3D

$$
\begin{array}{llcl}
\text{Skalarfeld:} & f & = & f(\mathbf{r}), \qquad \mathbf{r} = (x, y, z) \\
\text{Vektorfeld:} & \mathbf{A} & = & (A_x, A_y, A_z), \qquad A_i = A_i(\mathbf{r}) \\
\text{Nabla:} & \nabla & = & \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \; = \; \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \\
\text{Gradient:} & \nabla f & = & \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\text{Divergenz:} & \nabla \cdot \mathbf{A} & = & \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\text{Rotation:} & \nabla \times \mathbf{A} & = & \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
\text{Laplaceoperator:} & \Delta f & = & \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \; = \; \nabla \cdot \nabla f
\end{array}
$$

$$
\begin{array}{llcl}
\text{Eigenschaften:} & \nabla \times \nabla & = & 0 \\
 & \nabla \cdot (\nabla \times \mathbf{A}) & = & 0 \\
 & \nabla \times (\nabla \times \mathbf{A}) & = & \nabla(\nabla \cdot \mathbf{A}) - \nabla \cdot \nabla \, \mathbf{A} \\
 & \nabla \cdot \mathbf{r} & = & 3, \qquad \nabla \times \mathbf{r} = 0 \\
 & \nabla f(r) & = & \frac{\mathbf{r}}{r} \frac{\partial}{\partial r} f, \qquad r = |\mathbf{r}| \\
 & \nabla \cdot (\mathbf{r} f(r)) & = & 3f + r \frac{\partial f}{\partial r} \\
 & \nabla \times (\mathbf{r} f(r)) & = & 0
\end{array}
$$

## 20.3 Sphärische Koordinaten $r, \vartheta, \varphi$

Definition:
$$x = r\sin(\vartheta)\cos(\varphi)$$
$$y = r\sin(\vartheta)\sin(\varphi)$$
$$z = r\cos(\vartheta)$$

Basis:
$$\mathbf{e}_r = \begin{pmatrix} \sin(\vartheta)\cos(\varphi) \\ \sin(\vartheta)\sin(\varphi) \\ \cos(\vartheta) \end{pmatrix}, \quad \mathbf{e}_\vartheta = \begin{pmatrix} \cos(\vartheta)\cos(\varphi) \\ \cos(\vartheta)\sin(\varphi) \\ -\sin(\vartheta) \end{pmatrix}, \quad \mathbf{e}_\varphi = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

Gradient:
$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + \mathbf{e}_\varphi \frac{1}{r\sin(\vartheta)}\frac{\partial}{\partial \varphi}$$

Laplace:
$$\Delta = \Delta_r + \frac{1}{r^2\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\sin(\vartheta)\frac{\partial}{\partial\vartheta} + \frac{1}{r^2\sin^2(\vartheta)}\frac{\partial^2}{\partial\varphi^2}$$
$$\Delta_r = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}r = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

Volumen:
$$d^3r = r^2 dr d^2\Omega$$

Oberfläche:
$$\mathbf{d^2F} = d^2\Omega\, r^2\, \mathbf{e}_F, \qquad \mathbf{e}_F = \mathbf{e}_r$$

Raumwinkel:
$$d^2\Omega = d\varphi\, d(\cos\vartheta), \qquad \varphi \in [0, 2\pi), \qquad \vartheta \in [0, \pi]$$
$$\int d^2\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta\, \sin\vartheta = \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\vartheta)$$

## 20.4 Zylinderkoordinaten $r, \varphi, z$

Definition:
$$x = r\cos(\varphi)$$
$$y = r\sin(\varphi)$$
$$z = z$$

Basis:
$$\mathbf{e}_r = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}, \qquad \mathbf{e}_\varphi = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}, \qquad \mathbf{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$\mathbf{e}_r \times \mathbf{e}_\varphi = \mathbf{e}_z$ & zyklisch

Gradient:
$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z}$$

Laplace:
$$\Delta = \Delta_r + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$
$$\Delta_r = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} = \frac{1}{\sqrt{r}}\frac{\partial^2}{\partial r^2}\sqrt{r} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

Volumen: $d^3r = dz\, d\varphi\, dr\, r$

Oberfläche: $\mathbf{d^2F} = dz\, d\varphi\, r\, \mathbf{e}_F, \qquad \mathbf{e}_F = \mathbf{e}_r$

## 20.5 Integration in 2D und 3D

Linienintegrale:

$$\int_S d\mathbf{r}\cdot\mathbf{A}(\mathbf{r}) = \int_S \Big(dxA_x + dyA_y + dzA_z\Big) = \int ds \left(\frac{dx}{ds}A_x + \frac{dy}{ds}A_y + \frac{dz}{ds}A_z\right)$$

Weg $S$ parametrisiert durch $\mathbf{r} = \mathbf{r}(s)\ ,\ s \in [a,b]$

$$\Longrightarrow \quad \int_S d\mathbf{r}\cdot\mathbf{A}(\mathbf{r}) = \int_a^b ds\frac{d\mathbf{r}}{ds}\cdot\mathbf{A}\left(\mathbf{r}(s)\right)$$

Oberflächenintegral:

$$\int_F \mathbf{d^2F}\cdot\mathbf{A} = \int_F d^2F\,\mathbf{e}_F\cdot\mathbf{A}$$

$F$ = Integrationsfläche, $d^2F$ =Oberflächenelement

$\mathbf{e}_F$ = Normalenvektor der Oberfläche

Volumenintegral:

$$\int_V d^3r\ f(\mathbf{r}) = \int_V dx \int dy \int dz\, f(x,y,z)$$

$V$ = Integrationsvolumen

Gaußscher Satz:

$$\int_V d^3r\,\nabla\cdot\mathbf{A} = \int_{\partial V} d^2F\mathbf{e}_F\cdot\mathbf{A}$$

$\partial V$ = Rand von $V$ = geschlossene Oberfläche

Stokesscher Satz:

$$\int_F d^2F\,\mathbf{e}_F\cdot(\nabla\times\mathbf{A}) = \oint_{\partial F} d\mathbf{r}\cdot\mathbf{A}$$

$\partial F$ = Rand von $F$ = geschlossene Linie

$$\Longrightarrow \quad \nabla\times\mathbf{A} = 0 \quad \longleftrightarrow \quad \int_{\partial F} d\mathbf{r}\cdot\mathbf{A} = 0$$

$$\longleftrightarrow \quad \int_{\mathbf{a}}^{\mathbf{b}} d\mathbf{r}\cdot\mathbf{A} \quad \text{unabhängig vom Weg} \quad \mathbf{a}\to\mathbf{b}$$

Greenscher Satz:

$$\int_V d^3r\left(f\nabla^2g - g\nabla^2f\right) = \int_{\partial V} d^2F\,\mathbf{e}_F\cdot(f\nabla g - g\nabla f)$$

## 20.6 "Heavy-side"-Funktion und Dirac'sche $\delta$-Distribution

"Heavy-side" $\quad \theta(x) = \begin{cases} 1 & \text{für} \quad x > 0 \\ 0 & \text{für} \quad x < 0 \end{cases}$

"$\delta$"

$$\delta(x) = \frac{d}{dx}\theta(x)$$

$$\delta(x) = \lim_{\epsilon\to 0}\delta_\epsilon(x), \qquad \text{z.B.} \quad \delta_\epsilon = \frac{\exp\left(-\frac{x^2}{\epsilon^2}\right)}{\epsilon\sqrt{\pi}}$$

$$\delta(x) = 0 \quad \text{für} \quad x \neq 0 \qquad \& \qquad \int dx\,\delta(x) = 1$$

$$\int dx'\,\delta(x-x')f(x') = f(x)$$

$\delta(g(x))$ $\quad x_0 : \quad g(x_0) = 0 = \text{einzige Nullstelle von } g, \qquad g' = \frac{dg}{dx}$

$$\Longrightarrow \quad \int dx\delta\left(g(x)\right) f\left(x\right) = \int \frac{dg}{g'}\delta(g)f\left(x(g)\right) = \frac{1}{g'(x_0)}f(x_0)$$

$$\longleftrightarrow \quad \delta\left(g(x)\right) = \frac{\delta(x-x_0)}{g'(x_0)}$$

**3D**

kartesisch:

$$\int d^3r'\,\delta^3(\mathbf{r}-\mathbf{r}')f(\mathbf{r}') = f(\mathbf{r})$$

$$\delta^3(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z')$$

sphärisch:

$$\delta^3(\mathbf{r}-\mathbf{r}') = \frac{\delta(r-r')}{r^2}\delta(\cos\vartheta-\cos\vartheta')\delta(\varphi-\varphi')$$

$$= \frac{\delta(r-r')}{r^2}\frac{\delta(\vartheta-\vartheta')}{\sin\vartheta}\delta(\varphi-\varphi')$$

Kugel:

$$\delta^2(\Omega-\Omega') = \delta(\cos\vartheta-\cos\vartheta')\delta(\varphi-\varphi'), \qquad \Omega \equiv (\vartheta,\varphi)$$

axial:

$$\delta^3(\mathbf{r}-\mathbf{r}') = \frac{\delta(r-r')}{r}\delta(\varphi-\varphi')\delta(z-z'), \qquad r = \sqrt{x^2+y^2}$$

## 20.7 Kugelfunktionen $Y_{lm}(\vartheta, \varphi)$

"Drehimpuls" $\hat{\mathbf{L}} = -\imath \mathbf{r} \times \nabla$

$$\hat{L}_z = -\imath \frac{\partial}{\partial \varphi} \quad , \quad \hat{L}_\pm = \hat{L}_x \pm \imath \hat{L}_y = e^{\pm \imath \varphi} \left( \pm \frac{\partial}{\partial \vartheta} + \imath \cot(\vartheta) \frac{\partial}{\partial \varphi} \right)$$

$$\Delta = \Delta_r - \frac{1}{r^2} \hat{\mathbf{L}}^2 \quad , \quad -\hat{\mathbf{L}}^2 = \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta} \sin(\vartheta) \frac{\partial}{\partial \vartheta} + \frac{1}{\sin^2(\vartheta)} \frac{\partial^2}{\partial \varphi^2}$$

Eigensystem

$$\boxed{\begin{array}{ll} \hat{\mathbf{l}}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}, & \hat{l}_z Y_{lm} = \hbar m Y_{lm} \\ l = 0, 1, 2, .. & m = -l, -l+1, ..., l-1, l \end{array}}$$

$$\Longrightarrow \quad \boxed{\Delta R(r) Y_{lm}(\vartheta, \varphi) = \left( \Delta_r - \frac{l(l+1)}{r^2} \right) R(r) Y_{lm}}$$

Basis $\{Y_{lm}\}$ ist Basis auf der Oberfläche der Einheitskugel

orthogonal: $\int d^2\Omega \, Y^*_{lm}(\vartheta, \varphi) Y_{l'm'}(\vartheta, \varphi) = \delta_{ll'} \delta_{mm'}$

vollständig: $\sum_{lm} Y_{lm}(\vartheta, \varphi) Y^*_{lm}(\vartheta', \varphi') = \delta\left(\cos(\vartheta) - \cos(\vartheta')\right) \delta\left(\varphi - \varphi'\right)$

Rekursion $Y_{l-l} = \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} \sin^l(\vartheta) e^{-\imath l \varphi}, \qquad Y_{l\, m+1} = \frac{1}{\hbar \sqrt{(l-m)(l+m+1)}} \hat{L}_+ Y_{lm}$

Additions-theorem $\sum_{m=-l}^{l} Y_{lm}(\Omega) Y^*_{lm}(\Omega') = \frac{2l+1}{4\pi} P_l(\cos \Theta), \qquad \cos \Theta = \frac{\mathbf{r} \cdot \mathbf{r}'}{r r'}$

Relationen $Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \vartheta), \quad P_l(1) = 1, \quad P_l(-1) = (-)^l, \quad Y^*_{lm} = (-)^l Y_{l-m}$

Beispiele

$$Y_{00} = \sqrt{\tfrac{1}{4\pi}}$$

$$Y_{10} = \sqrt{\tfrac{3}{4\pi}} \cos(\vartheta) \qquad Y_{1\pm1} = \mp \sqrt{\tfrac{3}{8\pi}} \sin(\vartheta) e^{\pm \imath \varphi}$$

$$Y_{20} = \sqrt{\tfrac{5}{16\pi}} (3\cos^2(\vartheta) - 1) \qquad Y_{2\pm1} = \mp \sqrt{\tfrac{15}{8\pi}} \cos(\vartheta) \sin(\vartheta) e^{\pm \imath \varphi}$$

$$Y_{2\pm2} = \sqrt{\tfrac{15}{32\pi}} \sin^2(\vartheta) e^{\pm 2 \imath \varphi}$$

$$\begin{aligned} \int d^2\Omega \, Y^*_{lm}(\Omega) \delta(\cos\theta - 1) \delta(\varphi) &= \delta_{m0} Y_{l0}(\vartheta = 0) \\ &= \delta_{m0} \sqrt{\frac{2l+1}{4\pi}} P_l(1) = \delta_{m0} \sqrt{\frac{2l+1}{4\pi}} \end{aligned}$$