Exercise Sheet #9

Deadline: 08.01.2024, 12:00h

Problem 1 (*Langevin Equation*) (10 points)

Consider the Langevin equation:

$$m\dot{v} = -\gamma m v + \eta(t) \tag{1}$$

where the $\eta(t)$ term denotes white noise:

$$\langle \eta(t)\eta(t')\rangle = Q\delta(t-t') \tag{2}$$
$$\langle \eta(t)\rangle = 0\ . \tag{3}$$

Show that, in thermal equilibrium, the following holds:

$$\langle v(t)\eta(t)\rangle = 2k_{\mathrm{B}}\gamma T\ , \tag{4}$$

using the equipartition theorem: $\frac{1}{2}m\langle v^2(t)\rangle = \frac{k_{\mathrm{B}}T}{2}$.

Problem 2 (*Itô and Stratonovich Stochastic Calculus*) (10 points)

When considering differential equations of the form (compare (3.57) in the lecture notes)

$$\dot{x}(t) = F(x(t)) + b(x(t))\xi(t), \tag{5}$$

where $\xi(t)$ is white noise, problems arise due to the fact that the noise term depends on the variable $x(t)$ through the function $b(\cdot)$. These problems become apparent when trying to integrate the equation, where integration of the noise term yields

$$\int_0^t b(x(t'))\xi(t')\,\mathrm{d}t'. \tag{6}$$

Trying to discretize this integral using time intervals $\Delta t = t_{i+1} - t_i$ we find

$$(6) \to \sum_i \hat{b}_i \Delta W_i,$$

with $\Delta W_i = W_{i+1} - W_i$ being an increment of the stochastic process underlying the white noise (W_t is called a *Wiener Process*). This discretization is in analogy to the usual Riemann integration:

$$\int_0^t f(t')\,\mathrm{d}t' \to \sum_i \hat{f}_i \Delta t.$$

Note, that the coefficients $\hat{b}_t$ are random variables, since they depend on $x(t)$. Unlike for regular functions, where the sum converges in the limit $\Delta t \to 0$ no matter the midpoint $\hat{f}_i$, we may get very different results in the stochastic case, depending on where we choose to evaluate $\hat{b}_i$.
Two popular choices were found by Itô and Stratonovich, respectively, where Itô chooses to evaluate $\hat{b}_i$ at the left endpoints of the time intervals Δt and Stratonovich chooses to evaluate them in the center of the time intervals Δt. These two choices, however, may yield different results. The goal of this problem is to illustrate this.
For that, consider the noisy population growth equation

$$\dot{N} = N(r + \gamma\xi(t)), \tag{7}$$

where N is the population size and $r, \gamma \in \mathbb{R}$.

(a) A particularity of Itô stochastic calculus is that the chain rule has to be modified, which is manifested in Itô's famous lemma. For now, we treat Stratonovich's choice, where this is not the case.

Solve (7) in the Stratonovich interpretation. Increments of white noise are given by $\mathrm{d}W_t = \xi(t)\mathrm{d}t$, where

$$\int_0^t \mathrm{d}W_t = W_t,$$

with W_t being a Wiener Process.

(b) As mentioned in part (a), the chain rule has to be modified in the Itô interpretation. Itô's lemma states that for a random process $y(t)$, where $y(t) = g(x(t))$ with $g(\cdot)$ being a twice differentiable function and $x(t)$ as defined by (5) it holds that

$$\mathrm{d}y = \frac{\partial g}{\partial t}\mathrm{d}t + \frac{\partial g}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 g}{\partial^2 x}(b(x(t)))^2\mathrm{d}t.$$

Using this, find the solution in the Itô interpretation.