Exercise Sheet #7
Deadline: 11.12.2023, 12:00h

Problem 1 (Time-Delayed Differential Equation) (5 points)
Consider a car following a truck. The driver of the car should accelerate or break depending on the velocity of the truck that is in front. As the driver has a certain non-zero reaction time $T > 0$, the acceleration of the car is given by

$$\dot{v}(t + T) = \alpha (v_{\text{truck}}(t) - v(t)),$$

where $v(t)$ is the velocity of the car, $v_{\text{truck}}$ is the velocity of the truck and $\alpha > 0$ is an acceleration parameter of the car.

Assume that the truck has the constant speed $v(t) \equiv v_0$. Analyze the solution for the velocity $\dot{x}(t)$ for this case. Study the stability of this solution as function of the parameters $\alpha$ and $T$.

Problem 2 (Lorenz System) (15 points)
(a) Analyze the fixpoints of the Lorenz system given by

$$\dot{x} = s(y - x)$$
$$\dot{y} = x(r - z) - y$$
$$\dot{z} = xy - bz$$

and their stability as a function of the Rayleigh number $r > 0$ with parameters $s = 10$ and $b = 8/3$ fixed. Make a sketch of the $x(r)$ bifurcation diagram.

**Hint:** You may calculate the eigenvalues numerically.

(b) Show analytically that all fixpoints are unstable for

$$r > r_H = s(s + b + 3)/(s - b - 1).$$

**Hint:** You do not necessarily need a full solution for the eigenvalue spectra, since you are only looking for this particular transition point.

(c) What kind of bifurcation does the system undergo at $r = r_H$ and what kind of dynamics are present in the system for $r > r_H$?
(d) Evaluate numerically the Poincare map of the Lorenz model with $\sigma = 10$ and $b = 8/3$, for:

- $r = 22$ (regular regime) and the plane $z = 21$,
- $r = 28$ (chaotic regime) and the plane $z = 27$.

Make a plot of the maps and mark the fixpoints. What is the behavior of the points in each map as they evolve? (particularly in relation to the fixpoints).