Exercise Sheet #6

Deadline: 04.12.2023, 12:00h

Problem 1 (*Circular Limit Cycle*) (10 points)

This problem will help you to gain a better understanding of limit cycle attractors. Consider the system

$$\ddot{x} + a\dot{x}(x^2 + y^2 - 1) + x = 0,$$

where $a > 0$.

(a) Find and classify all the fixpoints.

(b) Show that the system exhibits a circular closed orbit and find its amplitude and period.
Hint: Transform the system to polar coordinates.

(c) Determine the stability of the closed orbit from part (b). Specifically, show that the closed orbit is a limit cycle attractor.

(d) Argue that the found limit cycle is unique in the sense that there are no other periodic trajectories.

Problem 2 (*Period–3 Cycle in Logistic Map*) (10 points)

In chaotic dynamics that arise from a cascade of period doubling bifurcations, one always finds windows of regular motion with respect to the bifurcation parameter. The widest of such windows shows a period–3 cycle. Consider the logistic map as an example system to study the occurrence of period–3 cycles:

$$x_{n+1} = f(x_n) \tag{1}$$
$$f(x_n) = r x_n (1 - x_n)\,, \tag{2}$$

where the variable $0 \le x_n \le 1$ and the bifurcation parameter $0 \le r \le 4$.

(a) In order to exert a period–3 cycle, the third iterated map has to fulfill the fixpoint condition $f(f(f(x_n))) = x_n$, while the map itself does not $f(x_n) \ne x_n$. Plot the third iterated for different values of the bifurcation parameter and try to understand how the period–3 cycle evolves.

(b) Calculate the lower bound r_3 for the bifurcation parameter, for which a period–3 cycle can exist. Follow these steps:

- Consider the three points of the cycle $x_n \to x_{n+1} \to x_{n+2} \to x_n$ being ordered and substitute $x = x_n$, $y = x_{n+1}$, $z = x_{n+2}$. Then you should find three equations linking x, y, z.
- As a fourth condition you can use the fact that the derivative of the third iterated function $\mathrm{d}f(f(f(x)))/\mathrm{d}x = 1$ is unity as it touches the diagonal. Re–write that condition in terms of x, y, z using the chain rule.
- Re-write the four conditions found above by the following substitution:

$$A = r\left(x - \frac{1}{2}\right), \qquad B = r\left(y - \frac{1}{2}\right), \qquad C = r\left(z - \frac{1}{2}\right). \quad (3)$$

  You should find that the equations are invariant under the cycle permutation of $A \to B \to C \to A$.
- Use the resulting equations to find the solution for A, B, C and finally the value of r_3. Keep in mind that you have to exclude the case $x = y = z$.

(c) At the point $r = r_3$ the period–3 cycle comes into existence, but one has to show that for $r = r_3 + \Delta r$ the cycle is stable for small but finite $\Delta r > 0$. Therefore, approximate one of the tangent points $x(t)$ of the third iterated map (where it touches the diagonal $x_{n+1} = x_n$) by a quadratic function that is shifted down (up) by $\Delta r > 0$ ($\Delta r < 0$) touching at $\Delta r = 0$. Show that one of the intersections with the diagonal has a slope less than one (absolute value) for some range in $\Delta r > 0$.