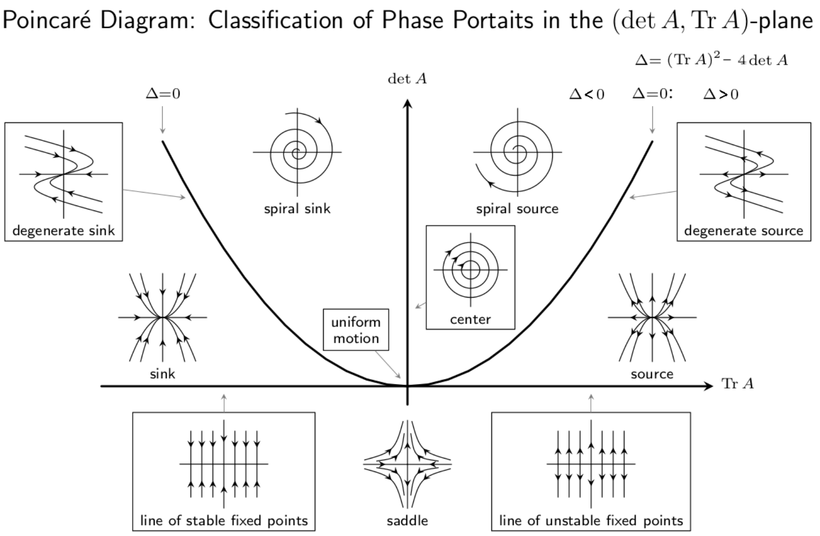
Exercise Sheet #5

Deadline: 27.11.2023, 12:00h

General Hint For the classification of fixpoints in 2D, the Poincaré diagram



is very helpful, where A is the stability matrix. Note that for a 2×2 matrix, the eigenvalues are in general given by

$$\lambda_{\pm} = \frac{1}{2}\,\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\,\mathrm{det}}.$$

Problem 1 (*Fixpoints and Bifurcations in 2D*) (10 points)

Consider the following two-dimensional dynamical system:

$$\dot{x} = x(a - 2x - y)$$
$$\dot{y} = a - x - 2y$$

where $a \in \mathbb{R}$ is a constant.

(a) Find all fixpoints of this system.

(b) Linearize the system near each fixpoint in order to determine their stability. What types of fixpoints are these for each value of a?

(c) Make a sketch of the flow in the system for $a > 0$, and a sketch for $a < 0$.

(d) What kind of bifurcation does this system exhibit when a changes sign?

Problem 2 (*Predator-Prey Model*) (10 points)

Consider a model for the interaction between predators (e.g. wolves) and preys (e.g. sheep),

$$\dot{x} = x\left[x(1-x) - y\right]$$
$$\dot{y} = y(x-a),$$

where $x \equiv x(t)$ is the number of sheep, $y \equiv y(t)$ is the number of wolves and $a \geq 0$ is a constant.

(a) Find all fixed points of the system and classify them.

(b) Show that the wolves go extinct if $a > 1$.

(c) A Hopf bifurcation occurs for a critical value a_c. Find a_c and compute the frequency of the limit cycle for a close to a_c.