Exercise Sheet #5

Problem 1 (*First Passage Time*)

Consider a random walk as described in the lecture Sec. 3.3.1. Let the probability density function of the walker's position at time t be given by the Gaussian distribution:

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

where D is the diffusion constant.

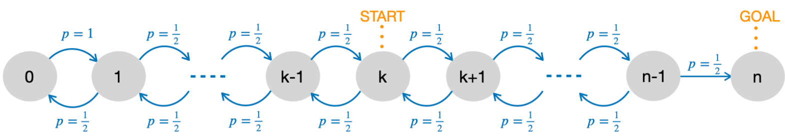
(a) Derive the distribution for the first passage time, (3.38) in the lecture notes:

$$F_y(t) = \frac{y}{\sqrt{4\pi D t^3}} \exp\left(-\frac{y^2}{4Dt}\right)$$

(b) The mean of this distribution diverges. Discuss that in the context of first passage times.

Problem 2 (*Bound Random Walk using Linear Difference Equations*)

Consider a random walker on a graph of length $n+1$, $n \gg 1$. The walker starts on node k and moves left and right with equal probability $p = 1/2$, where the boundary nodes $(0, n)$ only permit movement to the right or left, respectively. We want to compute the expected length of the random walk until the walker reaches the goal node n, denoted by $E_n^{(k)}$.



In lecture, you dealt with random walks by taking the continuum limit. Here we want to use *linear difference equations*.

(a) What are the boundary conditions $E_n^{(0)}$ and $E_n^{(n)}$?

(b) The general form of a second-order inhomogeneous linear difference equation is given by

$$z_{k+2} + a_1 z_{k+1} + a_2 z_k = r, \qquad r \in \mathbb{R}\,. \tag{1}$$

Write down a linear difference equation for the expectation value $E_n^{(k)}$.

(c) Solve the linear difference equation and prove

$$E_n^{(k)} = n^2 - k^2 \,.$$

Hint: The general solution of (1) is given by the superposition of the homogeneous equation, where we set $r = 0$, and a particular solution

$$z_k = z_k^{(h)} + z_k^* \,.$$

Make the ansatz $z_k = \lambda^k$ for the homogeneous part.