Exercise Sheet #3

Deadline: 13.11.2023, 12:00

Problem 1 (*Moments of Cluster Sizes*) (10 points)

Let the generating function of the degree distribution of a random graph be given by

$$G_0(x) = p_1 x + p_3 x^3 = (1 - p_3)x + p_3 x^3,$$

that is, a node can either have degree one with probability $p_1 = 1 - p_3$ or a degree three with probability p_3. Furthermore, as discussed in the lecture, we assume that the size of the graph is large enough so that we can ignore recurrent loops.

(a) Using the self-consistency equation derived in the lecture, find the analytic expression for $H_1(x)$, that is, the generating function for the size of clusters to a single node with one additional incoming connection. You should get it as the solution to a quadratic equation, which has two branches. Using the additional normalization constraint $H_1(1) = 1$, make sure to specify which solution to choose for a given value of p_3.

(b) Calculate the first and second derivative of your $H_1(x)$.

(c) Identify the Percolation threshold for the parameter p_3. Verify that both first and second moments of the distribution associated with $H_1(x)$ diverge at this threshold.

Problem 2 (*The Meaning of Scale-Free*) (10 points)

In this problem we want to understand the term *'scale-free'* graphs. For this we want to consider exemplarily a normalized scale-free degree distribution, the Tsallis–Pareto distribution,

$$p_k \approx p_{\mathrm{TP}}(k) = a(1+k)^{-1-a},$$

where $a > 0$.

(a) Show that $p_{\mathrm{TP}}(k)$ is indeed normalized.

(b) Next, we want to calculate the size of the largest hub in the graph, where a hub is a node with a high degree. Let k_{max} be the expected size of the

largest hub in the graph. In a graph of N nodes, you may assume to find at most one node with degree $k \in (k_{\max}, \infty)$, i.e. the probability to find such a node is $1/N$. From this assumption, find how $k_{\max}$ scales with N.
Hint: Express the probability to find a node with degree $k \in (k_{\max}, \infty)$ as an integral over the degree distribution.

(c) Compute the average degree $\langle k \rangle$ found in a scale-free graph with degree distribution $p_{\mathrm{TP}}(k)$, where $N \gg 1$.

(d) Compute the second moment of the Tsallis-Pareto distribution,

$$\langle k^2 \rangle = \int_0^{k_{\max}} k^2 p_{\mathrm{TP}}(k)\, \mathrm{d}k.$$

Consider the case that $a < 2$ and compare the first and second moment for a random graph (with Poisson degree distribution) and the scale-free graph in the limit $N \to \infty$. In what sense does the scale-free graph lack a scale?
Hint: You may use a computer (e.g. WolframAlpha) to find the integral.