Exercise Sheet #2

Deadline: 06.11.2023, 12:00

Problem 1 (*Probability Generating Function*) (10 points)

Generating functions can be very useful for sums in processes with multiple random variables. To show this we will look at a dice game, where a player moves X steps on a board each turn according to the number rolled on the dice.

(a) Write down the generating function $G_X(z)$ of the probability to get the number X on a six-sided dice throw.

(b) During a game the player advances each turn according to a dice throw. The number of turns is N and the total sum of steps taken during the game is S_N. Derive the generating function $G_{S_N}(z)$ of S_N as a function of $G_X(z)$.

(c) Now assume N is a random variable with distribution p_N, which is generated by the function $G_N(z)$. Prove that $G_{S_N}(z)$ is now given by

$$G_{S_N}(z) = G_N(G_X(z))$$

Hint: Use the law of total expectation, $E(x) = E_y(E(x|y))$.

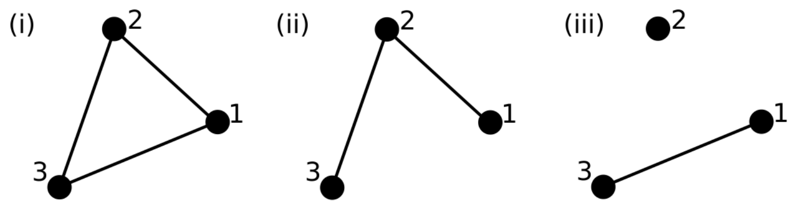
(d) Assuming that N is a poissonian variable with probability

$$p_N = e^{-\lambda}\frac{\lambda^N}{N!},$$

calculate the average number of steps in a game $\langle S_N \rangle$ using your previous result.

Problem 2 (*Graph Laplacian*) (5 points)

Consider the random graphs given below.



For each of these graphs, calculate

(a) the adjacency matrix and the normalized graph Laplacian.

(b) the eigenvalue spectrum of the Laplacian. What is its meaning?

Problem 3 (*Percolation*) (5 points)

We create a random graph by adding vertices with random edges connecting to other vertices. The degree of each vertex is geometrically distributed, decided by repeatedly tossing a coin until tails comes up, such that the degree is the number of heads tosses:

$$P(k) = (1-p)^{k-1}p$$

Assuming the coin is unfair with probability p for tails. Find the percolation threshold of the graph where a giant connected cluster forms.