Exercise Sheet #2

Problem 1 (*Homoclinic bifurcation*)

Consider the system

$$\ddot{x} = \mu\dot{x} + x - x^2 + x\dot{x}\,,$$

where $\mu \in \mathbb{R}$ is the bifurcation parameter. This system shows a homoclinic bifurcation, i.e. the collision of a saddle and a limit cycle. At the bifurcation point $\mu = \mu_c$, the limit cycle turns into a homoclinic orbit.

(a) Rewrite the system into a two-dimensional system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}\,.$$

(b) Solve the system numerically. Play around with the bifurcation parameter to numerically determine an estimate for the bifurcation point.

(c) Generate plots of the system below, at (or close to) and above the bifurcation point.

Problem 2 (*Hopf and limit cycle fold bifurcation*)

Consider the planar system in polar coordinates

$$\dot{r} = r(\mu + r^2 - r^4), \qquad \dot{\theta} = \omega + br^2,$$

where $\omega \neq 0$ and $b \in \mathbb{R}$ are fixed parameters and $\mu \in \mathbb{R}$ is the bifurcation parameter.

(a) Show that for $-1/4 < \mu < 0$ the system has a stable fixpoint, a stable and an unstable limit cycle.

(b) Show that the system undergoes a subcritical Hopf bifurcation at the Hopf bifurcation point $\mu_{\text{Hopf}} = 0$.

(c) Prove that a limit cycle fold bifurcation occurs. What is the bifurcation point?