C. Gros, D. Nevermann

Exercise Sheet #1

Problem 1 (*A spiral fixpoint spelled out*)

Consider the two-dimensional dynamical system in $\mathbf{x}(t) = (x(t), y(t)) \in \mathbb{R}^2$, $t \in \mathbb{R}$

$$\begin{aligned} \dot{x} &= ax + y \\ \dot{y} &= -x + ay \end{aligned} .$$

(a) Proof that $\mathbf{x}^* = (0, 0)$ is the only fixpoint of the system.

(b) Compute the Jacobian of the system evaluated at the fixpoint, $\mathbf{J}|_{\mathbf{x}^*}$.

(c) Compute the eigenvalues $\lambda_{1,2}$ and eigenvectors $\mathbf{v}_{1,2}$ of the Jacobian $\mathbf{J}|_{\mathbf{x}^*}$. Using Sec. 2.2.1 in the lecture notes, classify the fixpoint as a spiral.

Close to the fixpoint, we can approximate the full system by the linearization around the fixpoint

$$(\dot{\Delta \mathbf{x}}) = \mathbf{J}|_{\mathbf{x}^*} \, \Delta \mathbf{x}.$$

The general solution is given by a superposition of the eigenmodes

$$\Delta \mathbf{x} = C_1 \mathrm{e}^{\lambda_1 t} \mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t} \mathbf{v}_2 \,,$$

where $C_1 = \bar{C}_2$ for a real-valued solution.

(c) State the general solution for $\Delta \mathbf{x}$.
Hint: Use the identity

$$\mathrm{e}^{\pm \mathrm{i} t} = \cos t \pm \mathrm{i} \sin t \,.$$

(d) Plot the obtained solution in Python (let a chatbot help you, if you need it) and verify that the fixpoint is a spiral. What happens for different values of a?