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Project #2: Time-Delayed Chaos

Deadline: 15.06.2026, 12:00h

This project is about dynamical systems with time delay (compare Sec. 2.5 in the lecture notes). Dynamical systems with time delay are not written as ordinary differential equations, but instead described by delay differential equations (DDEs). For example, a system with a single discrete time delay τ is given by

$$\dot{x}(t) = F(t, x(t), x(t-\tau)), \qquad \tau > 0\,. \tag{1}$$

Notice, that now it does not suffice to specify a single initial value, as we typically do for ordinary differential equations. This is because at time $t = 0$, we need the value $x(-\tau)$ to compute the DDE and vice versa, to compute the solution $x(t \in [0, \tau])$, the DDE demands $x(t \in [-\tau, 0])$. This results in an infinite dimensional state space for even the simplest DDEs.

The additional complexity DDEs bring to the table goes hand in hand with enriched dynamics. Even for simple, single-variable equations one can observe complex dynamics such as chaotic dynamics. Chaos in time-delay systems will be the scope of this project.

Portrait of three time-delay dynamical systems

This section presents three time-delay dynamical systems which all show chaotic dynamics. All systems are defined in a single dynamical variable.

- **Mackey-Glass System:**

$$\dot{x}(t) = \frac{\alpha x(t-\tau)}{1 + (x(t-\tau))^{\gamma}} - \beta x(t)\,,$$

  where $\alpha, \beta, \gamma, \tau > 0$.

- **Dutta-Bhalekar System:**

$$\dot{x}(t) = -\gamma x(t) + k \sin\left(x(t-\tau_1)\right) - k\mathrm{e}^{-\gamma\tau_2} \sin\left(x(t-\tau_1-\tau_2)\right)\,,$$

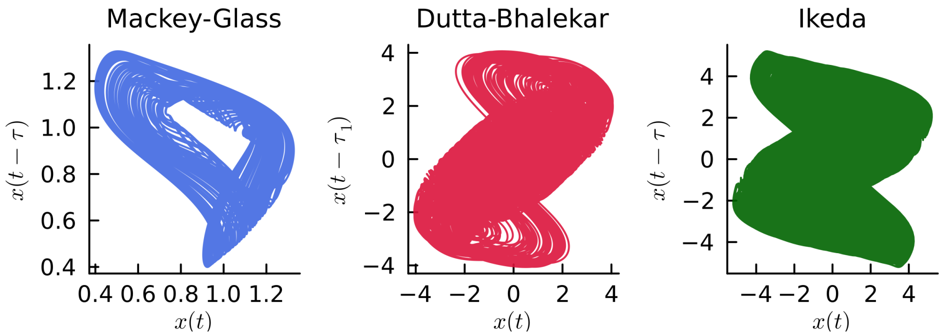
  where $k \in \mathbb{R}$ and the two discrete time delays are $\tau_1, \tau_2 > 0$.

- **Ikeda System:**

$$\dot{x}(t) = -\alpha x(t) + \beta \sin\left(x(t-\tau)\right)\,,$$

  where $\alpha, \tau > 0$.

The chaotic attractors of these three systems look like this:



In this, the parameter values are chosen such that the system is chaotic, and the systems are plotted in *stroboscopic projection*, i.e. in the projection $x(t)$-$x(t-\tau)$.

Task. *Choose one of the three systems above. A maximum of two students can choose a single system. You may also form groups of two students. We will deal with this in the tutorial session 27.05.2026. If you prefer to work on a different system, you are free to do that. It must be a system with time delay that exhibits chaos.*

Solving time-delay systems numerically

The numerical integration of dynamical systems with time delay works similar to systems without time delay. As a first step, the Euler method approximates the time derivative as

$$\dot{x} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \approx \frac{x(t+\Delta t) - x(t)}{\Delta t}\,.$$

Thus, the solution of (1) can be approximated by

$$x(t_{k+1}) = x(t_k) + F\left(t_k, x(t_k), x(t_k - \tau)\right) \Delta t\,,$$

where the solution is computed at discrete time steps $t_{k\in\mathbb{N}}$. The difficulty lies in the fact that the discretized solution space must contain past points $x(t_k - \tau)$. For the Euler method it suffices to choose time discretization as Δt such that the delay interval τ is exactly covered. More advanced numerical integration methods, such as RK4, usually require interpolation.

Task. *Figure out how to solve the system you chose numerically. You can, but do not have to, implement the numerical integrator yourself. There are packages for Python and Julia to integrate DDEs.*

Chaos in time-delay systems

There are several ways to identify chaos in (time-delay) dynamical systems. The most widely used metric is the maximal Lyapunov exponent $\lambda_{\max}$. Consider a pair of two close-by trajectories, separated by a distance $d(0) = \delta$. In a chaotic system, their respective trajectories are assumed to diverge exponentially

$$d(t) = \delta e^{\lambda_{\max} t},$$

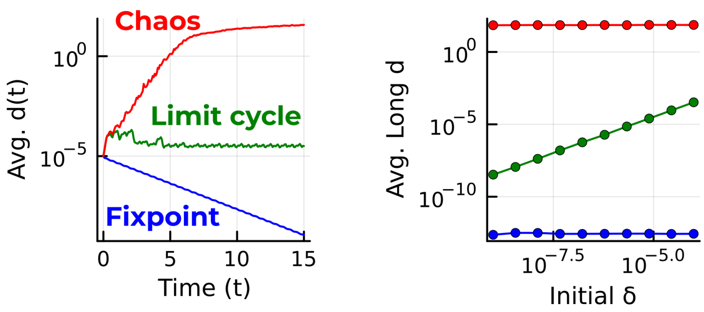
where

$$\lambda_{\max} = \lim_{t\to\infty} \lim_{\delta\to 0} \frac{1}{t} \log\left(\frac{d(t)}{\delta}\right).$$

For $\lambda_{\max} < 0$ system dynamics are regular, for $\lambda_{\max} > 0$ we get chaotic motion.
Another option is to consider cross-distance scaling. Consider an ensemble of pairs of initially close-by trajectories $x_1(t)$ and $x_2(t)$, then

$$d_{12}(t \gg T_\lambda) \propto \delta^\nu, \qquad d_{12} = \langle \|x_1(t) - x_2(t)\| \rangle,$$

where $d_{12}(0) = \delta$, ν is the distance scaling factor, T_λ is the time it takes x_1 and x_2 to reach a separation around the size of the chaotic attractor and $\langle\cdot\rangle$ denotes an ensemble average over the initial conditions. At time steps $t \gg T_\lambda$ one observes for regular motion that the distance between the trajectories stays roughly at its initial value δ and thus the scaling exponent $\nu \neq 0$. On the other hand, for chaotic motion, the trajectories diverge and end up spanning the whole chaotic attractor leading to $\nu \approx 0$. Below you see an example: On the left side d_{12} as a function of time, on the right side the cross-distance scaling for regular, chaotic motion and in the presence of a fixpoint.[1]



For more details on all of this, read the review [1].

[1] This plot was generated for the Lorenz system with $\beta = 8/3$, $\sigma = 10$ and $\rho = 10, 181.1, 180.7$ for fixpoint, limit cycle and chaos, respectively.

Task. *Discuss the system you chose. For example answer: Are there fixpoints? Where in the parameter landscape do we see chaotic / regular motion classified by one of the chaos measures? What does the attractor look like, and how does it change with the parameters? What is the fractal dimension of the attractor? . . .*
You don't have to tackle all these questions, but your discussion should give a concise overview what the dynamics of the system looks like.

Minimum requirements and format

The project is deliberately open – you choose the format and the details of your project. Nevertheless, there are some minimum requirements:

- Hand in the project by the deadline (15.06.26, 12:00h) via email to `nevermann@itp.uni-frankfurt.de`
- Present your project in the tutorial session on 17.06.26 and be prepared to answer questions.
- Hand-in a concise discussion of the system you chose (and the code you used to draw your conclusions). Discuss chaos in the system. You choose the format (e.g. written report, interactive HTML site, slides, notebook, . . .).

References

[1] Wernecke, H., Sándor, B., & Gros, C. (2019). Chaos in time delay systems, an educational review. arXiv preprint arXiv:1901.04826.