Exercise Sheet #10

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Problem 1 (*Random Walks*) 5 Pts

A drunken man is trying to walk along a straight sidewalk. He has already lost the notion of direction, so that he has an equal probability to make a step forward as back. Determine the displacement distribution after t steps, $P(x, t)$, its average displacement $\langle x \rangle$ (how far is he away from his staring point), and the variance $\langle x^2 \rangle - \langle x \rangle^2$ (the square of how much he actually walked) of its path. Find the exact result for $P(x, t)$ for a small number of steps $t \sim 10$. Find and an approximation for long walking time $t \to \infty$.

Problem 2 (*Fokker-Planck equation*) 5 Pts

Consider a random walk in one dimension, where instead of equally probable steps in both directions, you have:

- probability of a step to the left q
- probability of a step to the right $1 - q$

(a) Write down the master equation for the probability $P(x, t)$ to be at position x at time t.

(b) Suppose now, that steps $\mathrm{d}x \to 0$ are infinitesimal and we are in the continuous time limit $\mathrm{d}t \to 0$. Derive a differential equation for the time evolution of the probability $P(x, t)$.
Hint: You should get the Fokker-Planck equation in the general form.

(c) Find the corresponding expression for an equilibrated (*standard*) random walk.

Problem 3 (*Solution of Fokker-Planck equation in a harmonic potential with additional constant drift*) 5 Pts

The Fokker-Planck equation describes the evolution of a probability density function $P(x, t)$ under diffusion and balistic transport

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(v_\mathrm{d} P(x,t)\right) + \frac{\partial^2}{\partial x^2}\left(D P(x,t)\right) ,$$

with a drift velocity v_d and the diffusion coefficient D. Assume that

(1) the density distribution is stationary $\partial P(x,t)/\partial t = 0$,

(2) the drift velocity $v_\mathrm{d} = v_o + \frac{F(x)}{\gamma m}$ is given by a force $F(x) = -\mathrm{d}V(x)/\mathrm{d}x$ in a potential V (with m, γ denoting here the mass and the friction coefficient) and an additional constant velocity v_o,

(3) the potential $V(x) = fx^2/2$ is harmonic.

Calculate the solution of the Fokker-Planck equation.

Problem 4 (*Pareto distribution*) 5 Pts

The Pareto Type I distribution defined by

$$p(x) = \begin{cases} \alpha\, x_\mathrm{m}^\alpha\, x^{-\alpha-1} & \text{for } x \ge x_\mathrm{m} \\ 0 & \text{for } x < x_\mathrm{m} \end{cases}$$

where $x_\mathrm{m} > 0$ is the minimum possible value of for a random variable x, and α is a positive parameter. When this distribution is used to model the distribution of wealth, then the parameter α is called the *Pareto index*.

(a) Calculate the "raw" moments $\mu_n = \int_\mathbb{R} \mathrm{d}x\, p(x)\, x^n$ of the distribution. Check especially the three lowest moments with $n \in \{0,1,2\}$ and the variance.

(b) Calculate the median of the Pareto distribution and compare it to the mean.

(c) Calculate the cumulative probability distribution corresponding to the Pareto distribution.