Exercise Sheet #7

Hendrik Wernecke <wernecke@th.physik.uni-frankfurt.de>
Philip Trapp <trapp@th.physik.uni-frankfurt.de>

Problem 1 (*Catastrophic superlinear growth*) 6 Pts

The growth of a resource variable $x(t) \geq 0$, as defined by

$$\dot{x} = x^{\beta} - \gamma x, \qquad \gamma \in \{0, 1\} \tag{1}$$

is sublinear for $\beta < 1$ (superlinear for $\beta > 1$).

(a) Solve Eq. (1) analytically for super-linear growth $\beta > 1$ and $\gamma = 0$. Is there a singularity, viz a finite time $t_{\mathrm{s}} < \infty$ for which the limit $|\lim_{t \to t_{\mathrm{s}}} x(t)| = \infty$ diverges catastrophically? (2 Pts)

(b) Now discuss the behavior of the solution to Eq. (1) for $\beta > 1$ and $\gamma = 1$. (2 Pts)

(c) Consider sublinear decay $\beta < 1$ of Eq. (1) with $\gamma = 0$. Can the stable fixpoint be reached in finite time? (2 Pts)

Problem 2 (*Time-Delay Differential Equations*) 6 Pts

Consider a car following a truck. The driver of the car should accelerate or break depending on the velocity of the truck that is infront. As the driver has a certain non-zero reaction time $T > 0$ the accelaration of the car is given by:

$$\ddot{x}(t + T) = \alpha(v(t) - \dot{x}(t)) \ ,$$

where v is the velocity of the truck and $\alpha > 0$ represents the response of the motor.

Assume that the truck has the constant speed $v(t) \equiv v_o$. Analyse the solution for the velocity $\dot{x}(t)$ for this case. Study the stability of this solution as a function of the parameters α and T.

Problem 3 (*Reducible Time-Delay Differential Equations*) 8 Pts

Time-delay of dynamical systems can take different forms. Consider a dynamical system of the form

$$\dot{x}(t) = f\left(t, x(t), y(t)\right) \ ,$$

that depends on its current state $x(t) \in \mathbb{R}$ and by the term,

$$y(t) = \int_{-\infty}^{t} \mathrm{d}\tau\, x(\tau) g(\tau - t) = \int_{-\infty}^{0} \mathrm{d}\tau\, x(t+\tau) g(\tau)\,,$$

on its entire history, where $g(t) : \mathbb{R} \to \mathbb{R}$ is a suitable kernel function. This kind of time-delay is called continuous time-delay. An equation of this form is hard to solve, so one usually tries to re-write the system in terms of coupled first-order non-delay differential equations, e. g.:

$$\begin{aligned} \dot{x}(t) &= f\left(t, x(t), y(t)\right) \\ \dot{y}(t) &= g\left(t, x(t), y(t)\right)\,. \end{aligned}$$

Now consider a linear system of the form:

$$\dot{x}(t) = a x(t) - b y(t)\,, \tag{2}$$

with $a, b > 0$, and the moving (trailing) average of $x(t)$,

$$y(t) = \int_{-\infty}^{0} \mathrm{d}\tau\, x(t+\tau)\, \lambda \mathrm{e}^{\lambda\tau}\,,$$

with an exponential kernel, where $\lambda > 0$.

(a) Solve Eq. (2) using an exponential ansatz and determine the stability of the solution.

(b) Re-write this system in terms of a two-dimensional dynamical system of coupled non-delay differential equations. (*Hint*: Integration by parts might be useful.)

(c) Now investigate the two-dimensional system from dynamical systems point of view (i. e. fixpoints and stability). Compare the result to the previous result.