Exercise Sheet #6

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Problem 1 (*Attractors in gradient systems*) 4 Pts

In gradient systems the flow $\mathbf{f}(\mathbf{x}) : \mathbb{R}^N \to \mathbb{R}^N$ can be derived from a bifurcation potential $U(\mathbf{x}) : \mathbb{R}^N \to \mathbb{R}$,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = -\nabla U(\mathbf{x}) \ .$$

Show that gradient systems cannot have limit cycle attractors.

Problem 2 (*Derivative of sigmoidal*) 2 Pts

Calculate the partial derivatives of the sigmoidal function

$$y(x,b) = \frac{1}{1+\mathrm{e}^{a(b-x)}} \ ,$$

where a is a parameter. Prove that they hold

$$\frac{\partial y}{\partial x} = -\frac{\partial y}{\partial b}$$

and express them in terms of the original sigmoidal function.

Problem 3 (*Catastrophe*) 10 Pts

The membrane potential x of a single self-coupled neuron can be described by the dynamical system

$$\dot{x} = -x + y(x,b,a) \ , \tag{1}$$

where $a > 0$ and $b \in (0,1)$ are parameters describing, e. g., the concentration of ions in the neuron and y denoting the transfer function

$$y(x,b,a) = \frac{1}{1+\mathrm{e}^{a(b-x)}} \ . \tag{2}$$

The transferfunction describes the reaction of a neuron to the input x being in a certain state (x,b,a).

(a) Find the fixpoints of Eq. (1) depending on the parameters b, a. Plot the fixpoint manifold $\dot{x} = 0$ in the $b - x$ plane for a fixed $a > 4$. (2 Pts) (*Hint*: Find an expression for $b(x)$, where $\dot{x} = 0$.)

(b) Examine the stability of the fixpoint manifold. Plot the flow in the $b - x$ plane (again for a fixed $a > 4$). Can you identify any bifurcation? (By visual inspection, no proof needed.) (2 Pts)

(c) How does the manifold $\dot{x} = 0$ change qualitatively with a, especially at $a = 4$? Sketch the manifold for $a > 4$, $a = 4$ and $a < 4$ indicating the flow. (2 Pts) (Optional: Sketch the the manifold in the three dimensional $b - x - a$ space.)

Now assume that the parameter b evolves slowly in time (a is still fixed)

$$\dot{b} = \epsilon_b a \left[2y(x, b, a) - 1\right] , \tag{3}$$

where the time scale $0 < \epsilon_b \ll 1$ is finite but small.

(d) Considering the two dimensional dynamical system formed by Eqs. (1) and (3) in (x, b), find the fixpoint(s) of the system and determine the stability. (2 Pts)

(e) Sketch a few trajectories in the $x - b$ plane for small ϵ_b, i. e. the influence of the manifold $\dot{x} = 0$ is much stronger than the time evolution $\dot{b}$ of the parameter. What attractor would you expect to find and how does it change when varying ϵ_b? (2 Pts)

Problem 4 (*Lorenz system*) 4 Pts

Analyse the fixpoints and their stability of the Lorenz system:

$$\begin{aligned}
\dot{x} &= -\sigma(x - y) \\
\dot{y} &= -xz + rx - y \\
\dot{z} &= xy - bz
\end{aligned}$$

for the $r > 0$, $\sigma = 10$ and $b = 8/3$ parameters, and make a sketch of the $x^*(r)$ bifurcation diagram. (*Hint*: You may calculate the the eigenvalues numerically.)