Exercise Sheet #5

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Problem 1 (*Heteroclinic orbit*) 6 Pts

Consider the two-dimensional dynamical system defined by:

$$\begin{aligned} \dot{x} &= 1 - x^2 \\ \dot{y} &= yx + \epsilon\,(1 - x^2)^2\,. \end{aligned}$$

Find the fixpoints of the system, check their type and stability by calculating the eigenvalues of the Jacobian. Calculate the eigenvectors corresponding to the eigenvalues. Sketch the flow in phase space including the stable and unstable manifolds for $\epsilon = 0$. How does the sketch change for small but finite $\epsilon > 0$?

Problem 2 (*Bifurcation on invariant cycle*) 6 Pts

The following system describes a limit cycle intersected by a the line $x = a$:

$$\begin{aligned} \dot{x} &= -y(x-a) - x(x^2+y^2-1) \\ \dot{y} &= x(x-a) - y(x^2+y^2-1)\,, \end{aligned}$$

with a real parameter a.

(a) Find all fixpoints of the system and determine their stability. (2 Pts)

(b) Sketch the flow in phase space for $|a| < 1$, $|a| = 1$, and $|a| > 1$. (2 Pts)

(c) Rewrite the system in polar coordinates. (2 Pts)

Problem 3 (*Triangle map*) 8 Pts

Consider the map $x_{t+1} = f(x_t)$ with

$$f(x) = \begin{cases} rx & \text{if } 0 \le x < 1/2 \\ r - rx & \text{if } 1/2 \le x \le 1 \end{cases}\,,$$

for $0 \le x \le 1$ and the parameter $0 \le r \le 2$.

(a) Plot the function. (2 Pts)

(b) Look for fixed points and cycles. (2 Pts)

(c) Derive the analytic expression for the maximal Lyapunov exponent, defined by

$$\lambda_{\max} = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\mathrm{d}f^{(n)}(x)}{\mathrm{d}x} \right| \;, \text{ where } \; f^{(n)}(x) = f\left(f^{(n-1)}(x)\right) \;.$$

Hint: Use the chain rule. (2 Pts)

(d) For which range of r does the triangle map exhibit chaos? (2 Pts)